THE UNKNOTTING NUMBER AND BAND-UNKNOTTING NUMBER OF A KNOT

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Lemma 2.7. *Let P be a reduced knot projection. Then, $tr(P) = p(P) - 2$ if and only if P is one of the projections of positive or negative 3-braid knot diagrams as illustrated in* Fig. 3 *and the projections of the connected sum of a $(2, r)$-torus knot diagram and a $(2, s)$-torus knot diagram for some odd integers $r, s \neq \pm 1$.*

Proof. First, we show the 'if' part. If P is one of the projections of the connected sum of a $(2, r)$-torus knot diagram and a $(2, s)$-torus knot diagram, it follows from Theorem 2.6 and Proposition 2.2 that $tr(P) = p(P) - 2$. Suppose that P is one of the projections of positive 3-braid diagrams. By Theorem 2.6, $tr(P) \leq p(P) - 2$. Assume that $tr(P) < p(P) - 2$. Let Q be a trivial pseudo diagram which realizes the trivializing number of P. Let $p_1, p_2, \ldots, p_n$ be the pre-crossings of Q. Then $n \geq 3$ since $tr(P) < p(P) - 2$. Let P' be the projection obtained from P by smoothing $p_1, p_2, \ldots, p_n$. Then P' is a projection of $(n + 1)$-component link diagram from Proposition 2.4. This contradicts that P is one of the projections of positive 3-braid knot diagrams.

Next, we show the 'only if' part. If P is not prime, P is the projection of the connected sum of a $(2, r)$-torus knot diagram and a $(2, s)$-torus knot diagram for some odd integers $r, s \neq \pm 1$ from Proposition 2.3, Theorem 2.6 and Proposition 2.2.

Suppose that P is prime. We show that one of the components of P_p is a projection of a $(2, t)$-torus knot diagram for some odd integer t and the other component of P_p has no self pre-crossings for any pre-crossing p where P_p is the projections obtained from P by smoothing p. Namely, for any chord d there exists a chord which does not cross d in CD_P. Let P_1 and P_2 be the knot projections of P_p. If each of P_1 and P_2 has no pre-crossings, this implies that $p(P)$ is odd. This contradicts that $tr(P)$ is even by Theorem 2.5. If each of P_1 and P_2 has a pre-crossing, this implies that $tr(P) < p(P) - 2$. Without loss of generality, we may assume that P_1 has a pre-crossing. If P_1 is not one of the projections of $(2, t)$-torus knot diagrams, $tr(P_1) < p(P_1) - 1$ by Theorem 2.6. This implies that $tr(P) < p(P) - 2$ and hence contradicts. Therefore, one of the components of P_p is the projection of a $(2, t)$-torus knot diagram for some odd integer t and the other component of P_p has no self pre-crossings for any pre-crossing p.

We suppose that P_1 is the projection of a $(2, t)$-torus knot diagram. Let p' be a pre-crossing of P_1 and P_1' and P_1'' the knot projections obtained from P_1 by smoothing p' such that P_1'' has the pre-crossing p in P. Note that each of P_1' and P_1'' does not have a pre-crossing. Let $a_1, a_2, \ldots, a_n$ (resp. $b_1, b_2, \ldots, b_m$) be the pre-crossings of P_1' (resp. P_1'') and P_2 which appear on P_2 from p in this order along the orientation. We show that $a_1, a_2, \ldots, a_n$ appear on P_1' from a certain point in this order along the orientation and also $b_1, b_2, \ldots, b_m$ appear on P_1'' from a certain point in this order along the orientation. The pre-crossings $b_1, b_2, \ldots, b_m$ appear on P_1'' from a certain point in this order along the orientation since one of the projections from P by smoothing p' is one of the projections of $(2, s)$-torus knot diagrams. Next, we show that the pre-crossings $a_1, a_2, \ldots, a_n$ appear on P_1'' from a certain point in this order along the orientation. Suppose that $n > 2$. If there exists a part of a chord diagram as illustrated in Figure 1 then this contradicts $tr(P) = p(P) - 2$ by Theorem 2.5.

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Therefore, we consider the four cases as illustrated in Figure 2. The chord diagrams of these cases are as in Figure 3. The point x joins x' at each case in Figure 2. Then, let q be a pre-crossing where an arc from x to x' crosses a bold line. If q is on P_1'', any two of the three chords corresponding to a_k, a_l and q do not cross in CD_P and this implies $tr(P) < p(P) - 2$. If q is on P_2, any two of the three chords corresponding to a_i, a_j and q do not cross in CD_P and this implies $tr(P) < p(P) - 2$. Thus, $a_1, a_2, \ldots, a_n$ appear on P_1' from a certain point in this order along the orientation. Therefore, P is one of the projections of positive or negative 3-braid knot diagrams. □

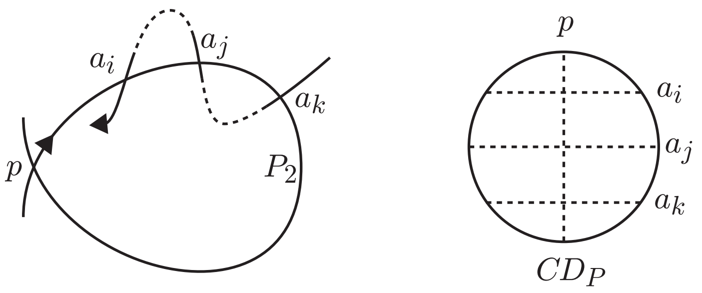

FIGURE 1

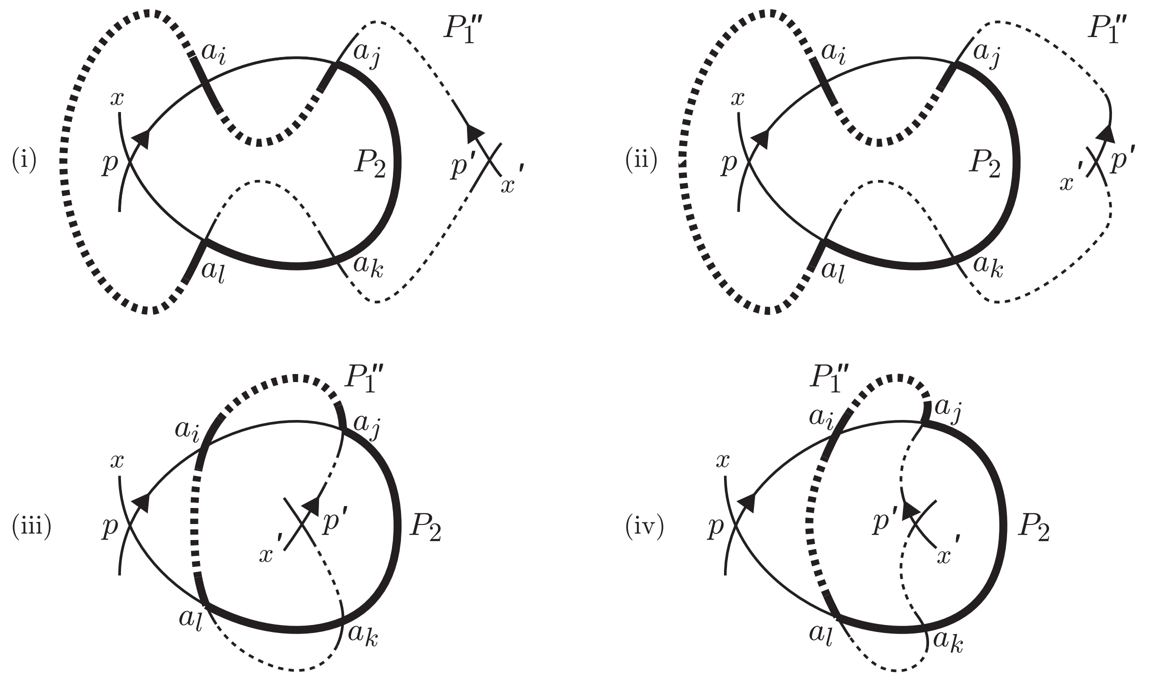

FIGURE 2

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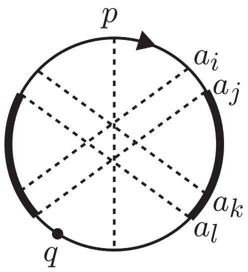

FIGURE 3

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