結び目，絡み目及び空間グラフの
準射影図と
その応用について

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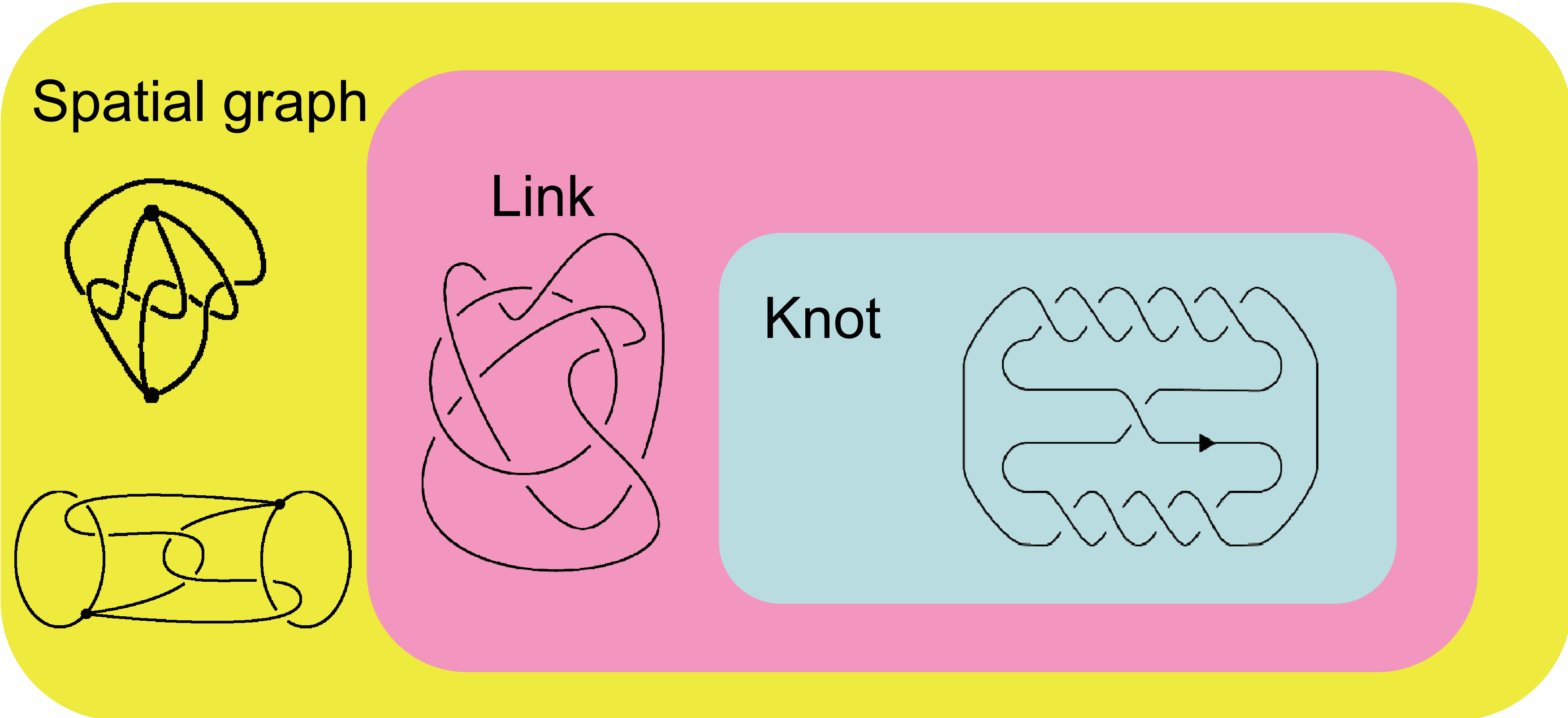
Contents

Knots, Links, Spatial Graphs

Knot : a circle embedded in $\mathbf{R}^3$ ($\mathbf{S}^3$)

Link : some circles embedded in $\mathbf{R}^3$ ($\mathbf{S}^3$)

Spatial graph : a graph embedded in $\mathbf{R}^3$ ($\mathbf{S}^3$)



Notation & Definition

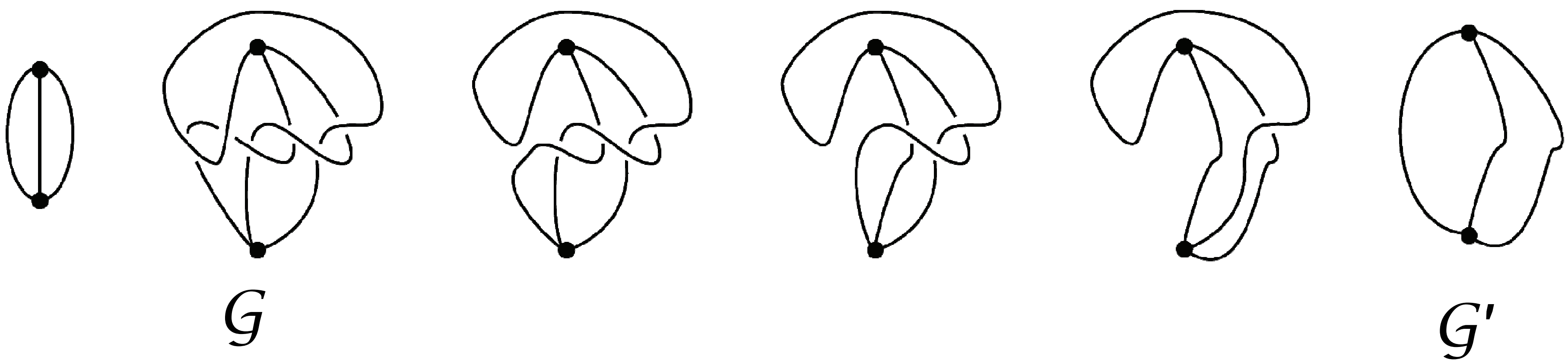
$\mathcal{G}, \mathcal{G}'$: spatial graphs

$\mathcal{G}$ and $\mathcal{G}'$ are equivalent ($\mathcal{G} \sim \mathcal{G}'$)

$\Leftrightarrow$ $\exists h : \mathbf{R}^3 \to \mathbf{R}^3$: orientation preserving self-homeo.

s.t. $h(\mathcal{G}) = \mathcal{G}'$

$\mathcal{G}$ is trivial (or unknotted) $\Leftrightarrow$ $\exists \mathcal{G}' \sim \mathcal{G}$ s.t. $\mathcal{G}' \subset \mathbf{R}^2 \subset \mathbf{R}^3$



Definition of Projection

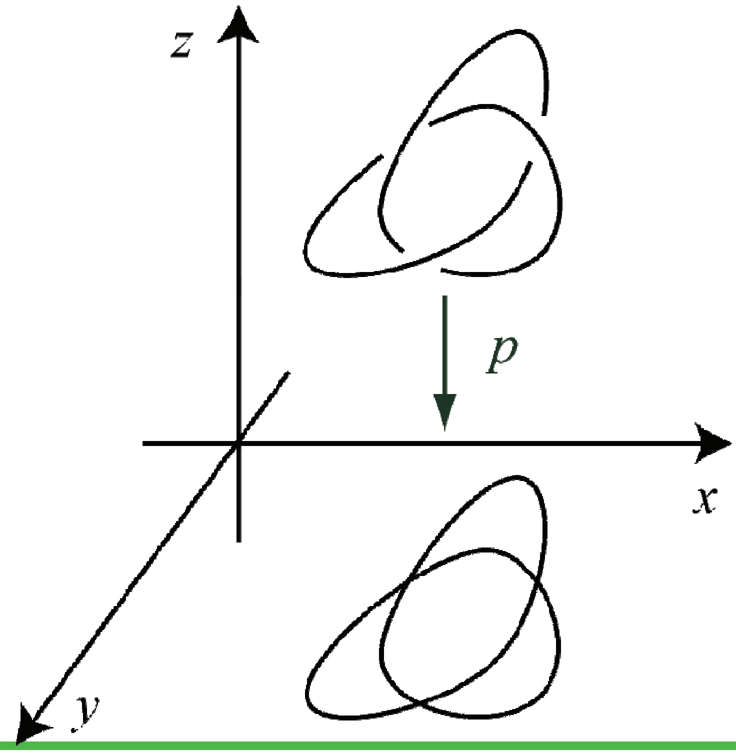
$p : \mathbf{R}^3 \to \mathbf{R}^2$: natural projection

p is a projection of a knot K (spatial graph)

$\Leftrightarrow$ multiple points of $p|_K$ are only finitely many transversal double points (away from the vertices).

We call $p(K)$ a projection

and denote it by $P = p(K)$.



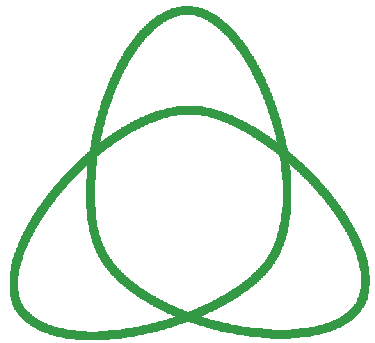
Definition of Diagram

A diagram *D* is a projection *P* with over/under information

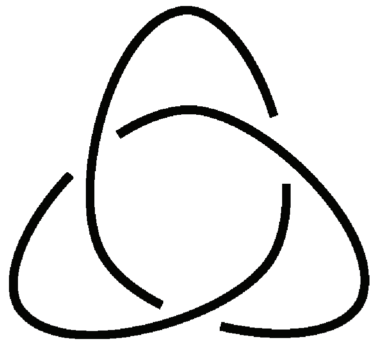
at every double point.

Then we say *D* is obtained from *P*.

A diagram uniquely represents a knot up to equivalence.



P



D

A Partial Order of Knots and Links

PROJ(L) : the set of all projections of an unoriented link L

L_1 is a minor of L_2 ($L_1 \leqq L_2$, $L_2 \geqq L_1$) [Taniyama'89]

$\Leftrightarrow$ PROJ(L_1) $\supset$ PROJ(L_2)

$\mathcal{L}^\mu$: the set of all μ-component links

Proposition [Taniyama '89]

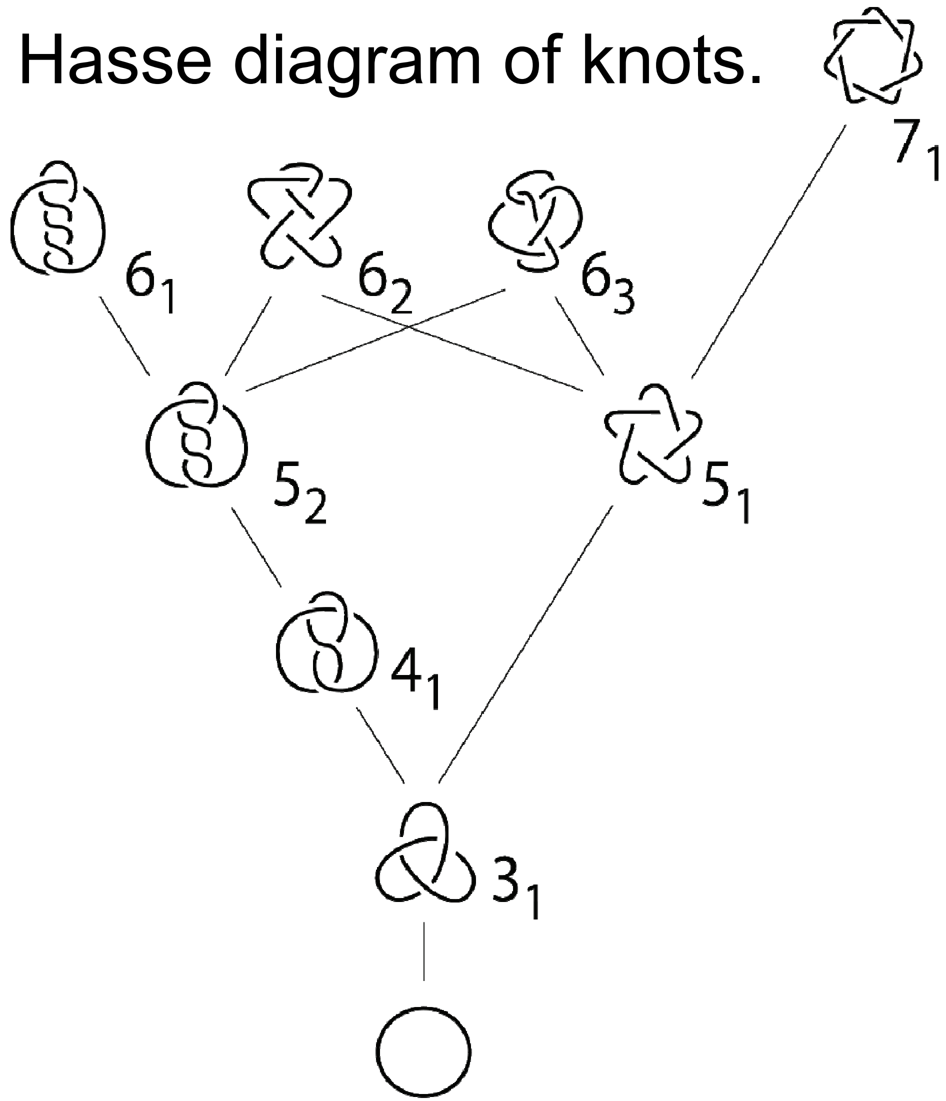
The pair ($\mathcal{L}^\mu$, $\leqq$) is a pre-ordered set for each natural number μ. Namely the following (1) and (2) hold for any L_1, L_2 and L_3 in $\mathcal{L}^\mu$.

(1) $L_1 \geqq L_1$ (reflexive law).

(2) $L_1 \geqq L_2$, $L_2 \geqq L_3 \Rightarrow L_1 \geqq L_3$ (transitive law).

A Partial Order of Knots and Links

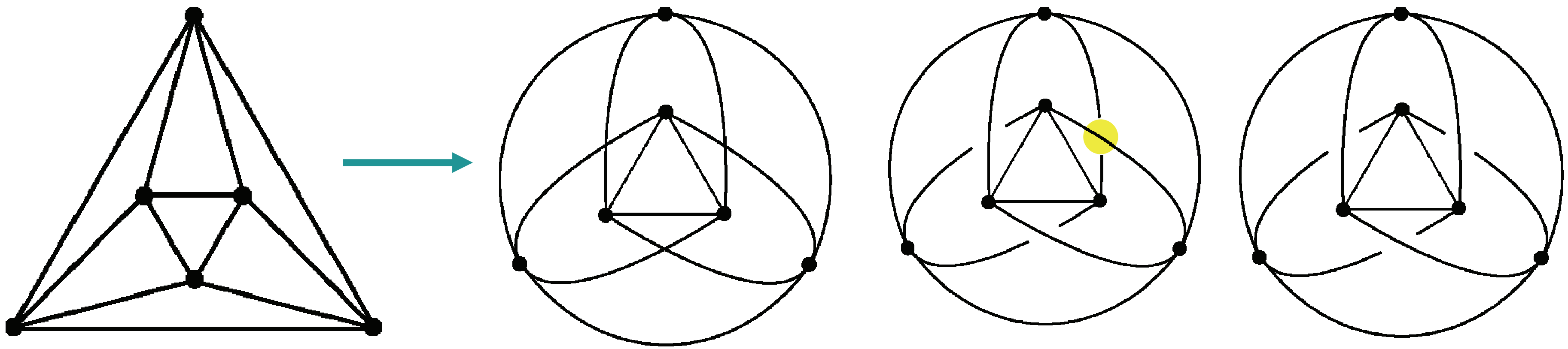
Taniyama has the Hasse diagram of knots.



Knotted Projection

A projection *P* is knotted [Taniyama '95]

⇔ Any diagram obtained from *P* represents a nontrivial spatial graph.

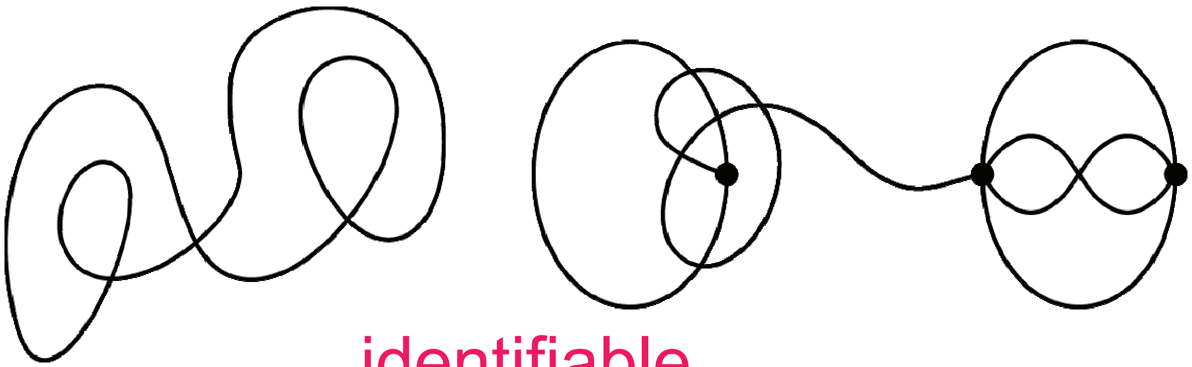


Graphs which have a knotted projection have not been characterized completely yet.

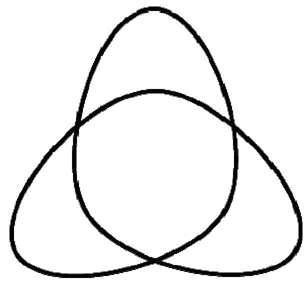
Identifiable Projection

A projection *P* is identifiable [Huh-Taniyama '04]

⇔ Any two diagrams obtained from *P* represent the same spatial graph as labeled spatial graphs.



identifiable



not identifiable

Theorem [Nikkuni '05]

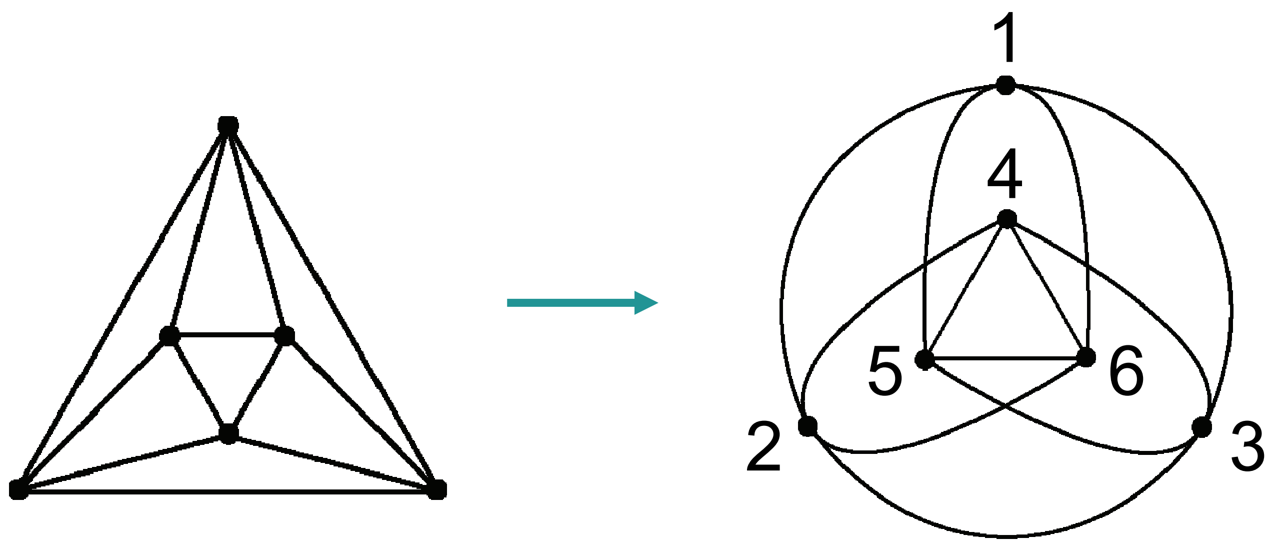
P is identifiable projection

⇒ Any diagram obtained from *P* represents a trivial spatial graph.

Completely Distinguishable Projection

A projection *P* is completely distinguishable [Nikkuni '06]

⇔ Any two different diagrams obtained from *P* represent different spatial graphs as labeled spatial graphs.

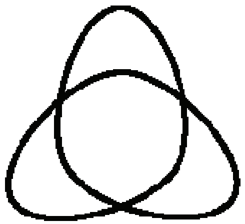


Motivation of Pseudo Diagram

Q. Can we determine from a projection whether the original spatial graph is trivial or knotted?

Ans. We cannot determine except some special cases.

Ex.



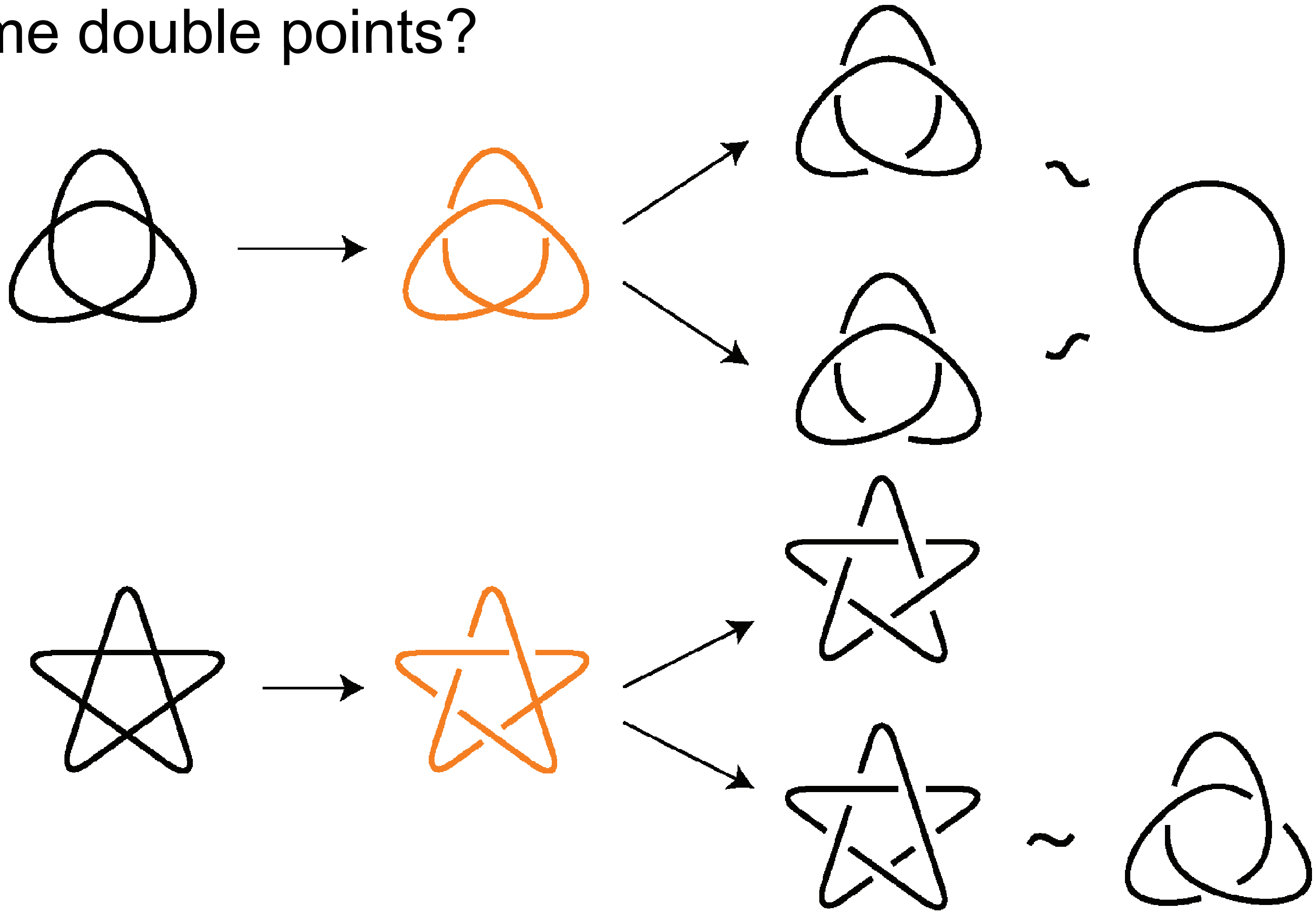
2^3 diagrams

Two diagrams represent nontrivial knots and six diagrams represent a trivial knot.

Motivation of Pseudo Diagram

How does it become if we know over/under information at some double points?

Ex.



Motivation of Pseudo Diagram

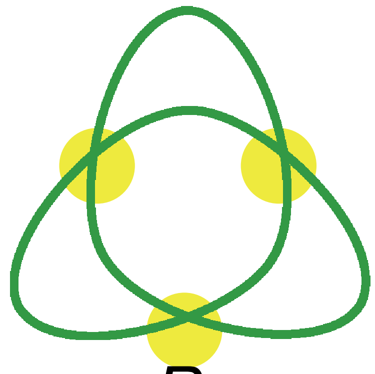
- Which double points of a projection and which over/under informations at them should we know in order to determine that the original knot is trivial or knotted?
- We introduced a notion of the pseudo diagram in [H, 2009].

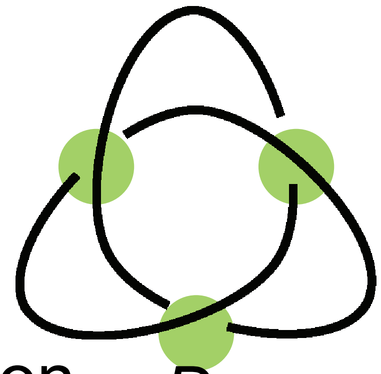
Definition of Diagram

A diagram *D* is a projection *P* with over/under information at every double point.

Then we say *D* is obtained from *P*.

A diagram uniquely represents a knot up to equivalence.





Then a double point with over/under information is called a crossing and a double point without over/under information is called a pre-crossing.

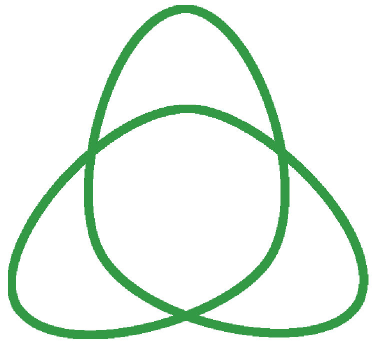
Definition of Pseudo Diagram

A pseudo diagram Q is a projection P with over/under information at some pre-crossings.

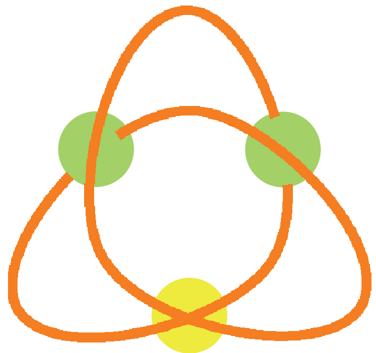
Thus, a pseudo diagram Q has crossings and pre-crossings.

Here, Q possibly has no crossings or no pre-crossings.

Namely, Q is possibly a projection or a diagram.



P



Q

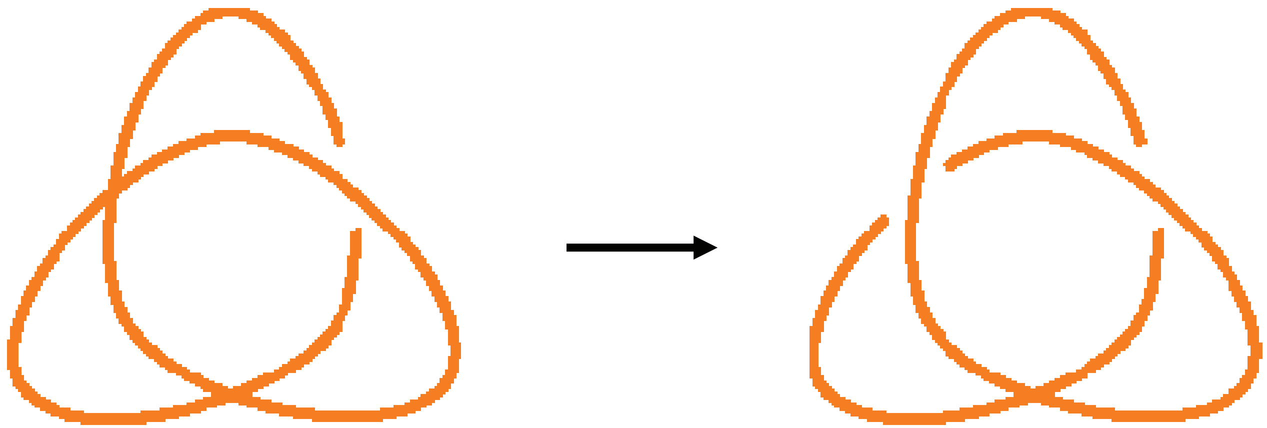
Relation between Pseudo Diagrams

Q, Q' : pseudo diagrams of a projection

A pseudo diagram Q' is obtained from a pseudo diagram Q.

⇔ Each crossing of Q has the same over/under information as Q'.

Ex.



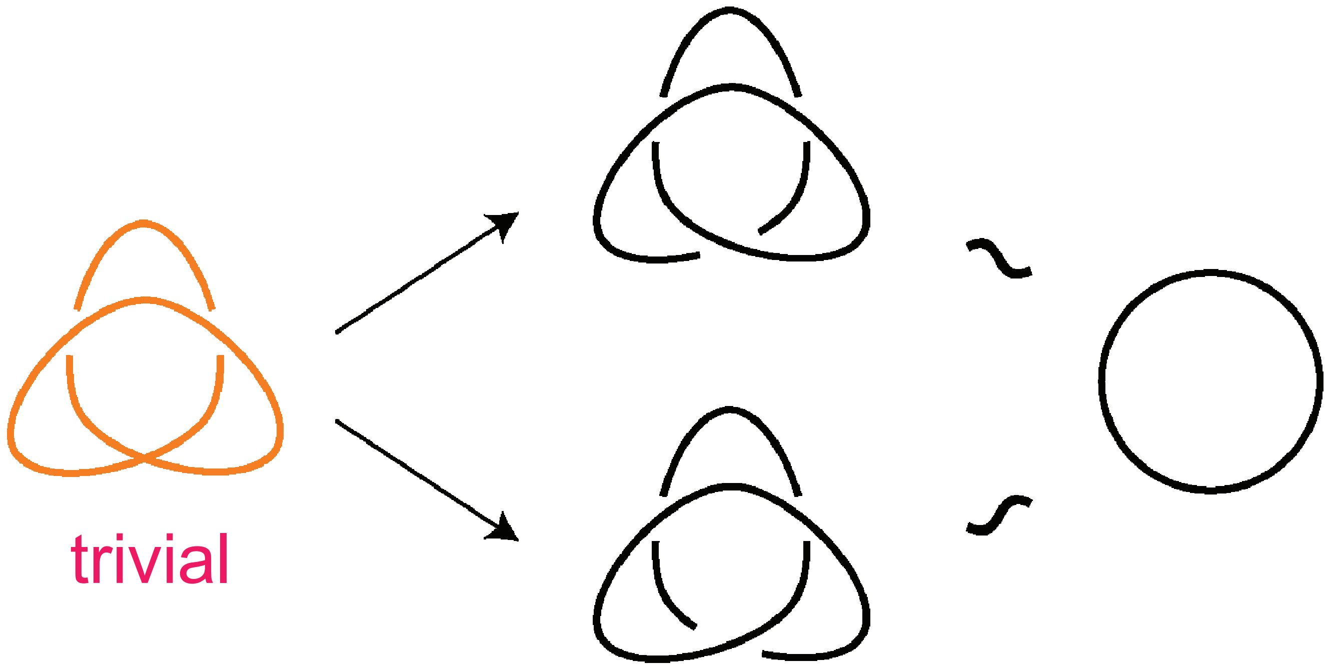
Q $\longrightarrow$ Q'

Triviality of Pseudo Diagram

A pseudo diagram Q is trivial.

⇔ Any diagram obtained from Q represents a trivial knot.

Ex.



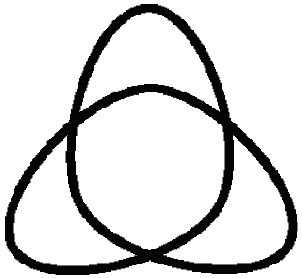
Trivializing Number

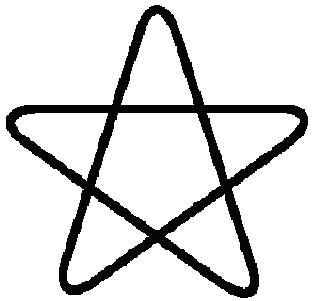
tr(P):＝min{ c(Q) | Q : trivial pseudo diagram obtained from P }

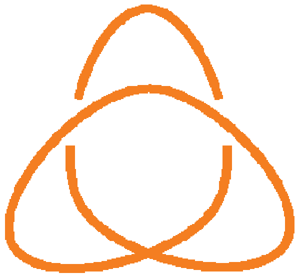
where c(Q) is the number of the crossings of Q.

We call tr(P) the trivializing number of P.

Ex.

tr()＝2

tr()＝4



Result 1 on Trivializing Number

Theorem 1-1

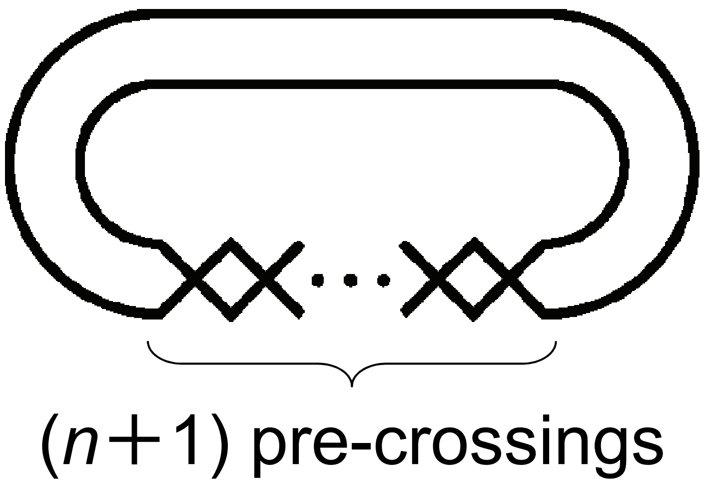
P : knot projection $\Rightarrow$ tr(P) is even.

Proposition 1-2

$\forall n$: even positive number

$\exists P$: knot projection with tr(P) = n

Ex.

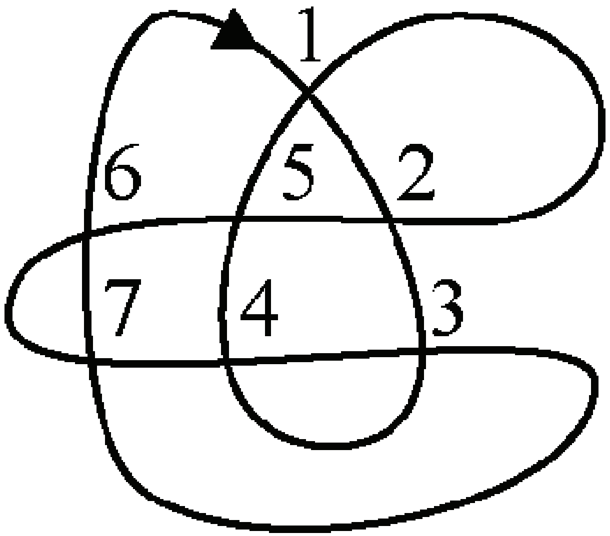


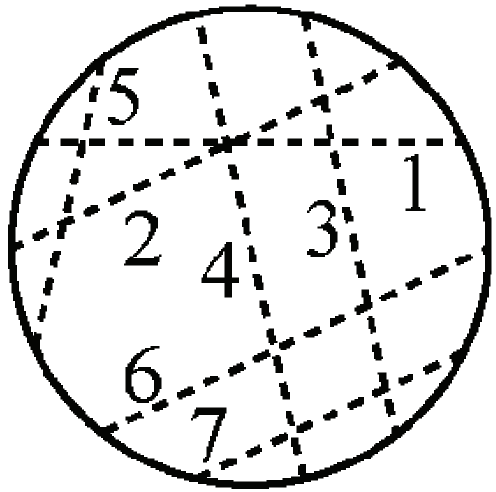
On Trivializing Number and Chord Diagram 1

Q : pseudo diagram of a circle with n pre-crossings

A chord diagram of Q is a circle with n chords marked on it by dashed line segment, where the pre-image of each pre-crossing is connected by a chord.

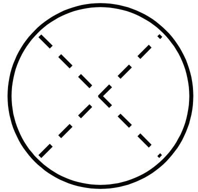
Ex.



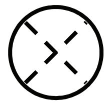


On Trivializing Number and Chord Diagram 2

Lemma 1-3

Q : pseudo diagram of a circle s.t. CD_Q contains 

$\Rightarrow$ Q is not trivial.

Proof. Q' : pseudo diagram obtained from Q with $CD_{Q'} =$ 

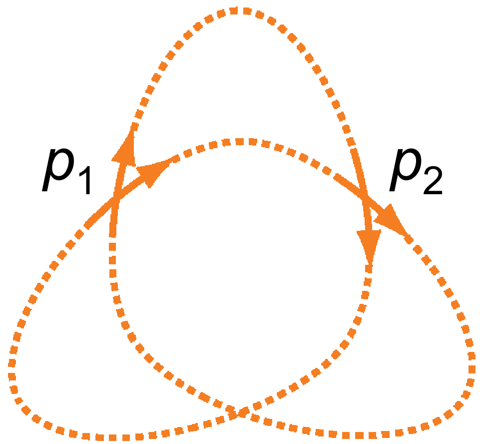
K_1 : the knot represented by D++

where ++ means that p_1 is + and p_2 is +

K_2 : the knot represented by D+−

K_3 : the knot represented by D−+

K_4 : the knot represented by D−−



$a_2(K_1) - a_2(K_2) - a_2(K_3) + a_2(K_4) = 1$

where a_2 is the second coefficient of the Conway polynomial.

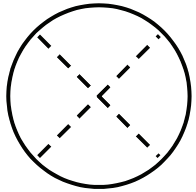
Therefore, a diagram representing a nontrivial knot is obtained from Q. ■

On Trivializing Number and Chord Diagram 3

Lemma 1-4

P : knot projection

CD : sub-chord diagram of CD_P

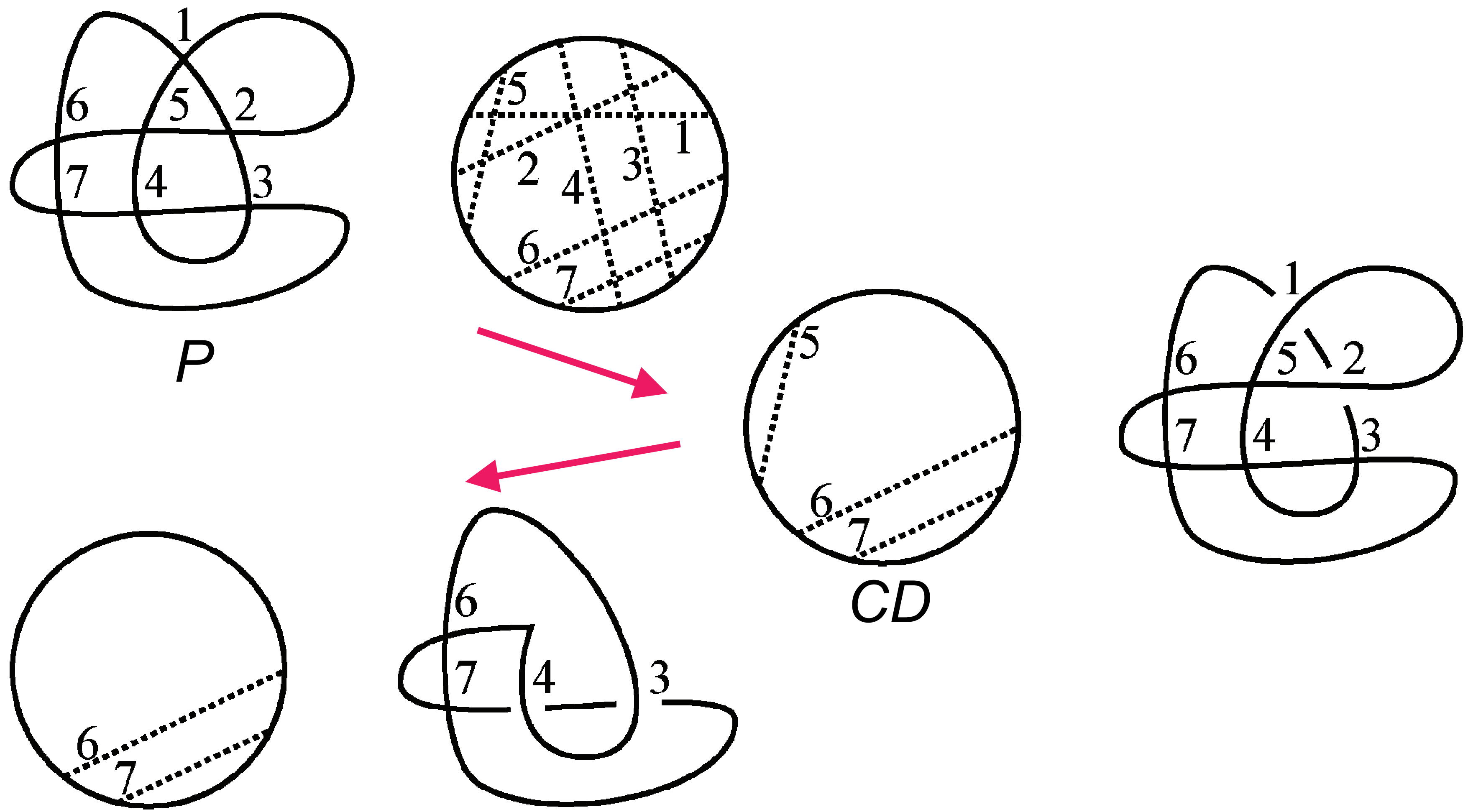
s.t. CD does not contain 

⇒ ∃ Q : a trivial pseudo diagram obtained from P

s.t. $CD_Q = CD$

Proof of Lemma 1-4

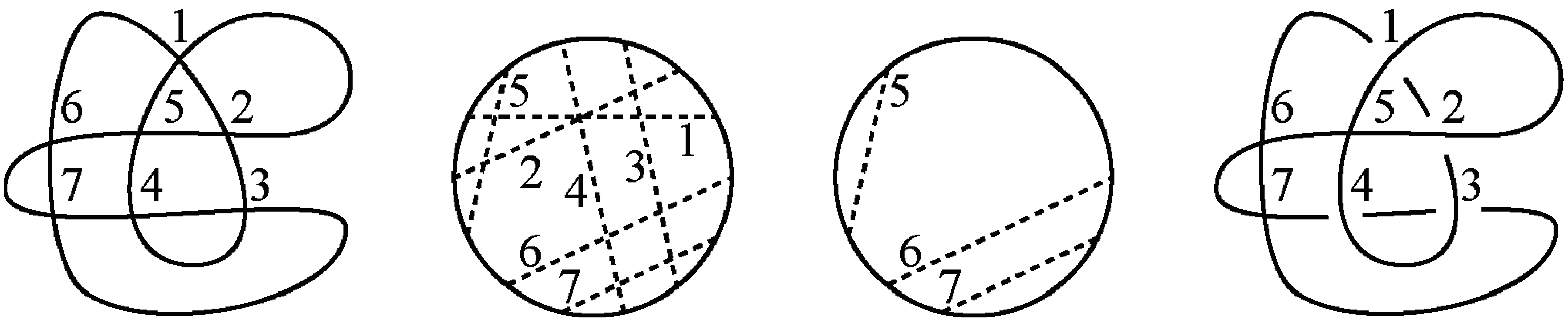
Sketch Proof of Lemma 1-4.



On Trivializing Number and Chord Diagram 4

For a knot projection P,
by applying Lemmas 1-3 and 1-4,
we can get tr(P) from CD_P.

Ex



Therefore, tr(P)=4

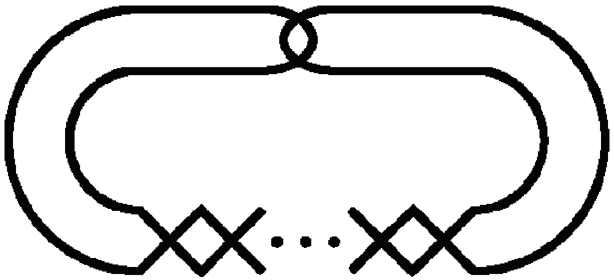
Result 2 on Trivializing Number

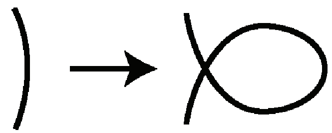
Theorem 1-5

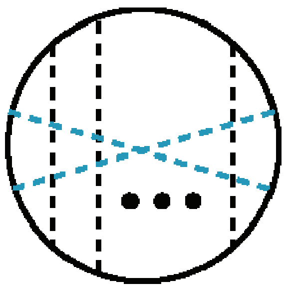
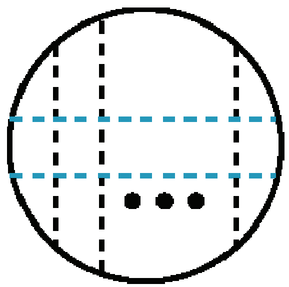
P : knot projection

tr(P)＝2

⇔ P is obtained from



by a series of replacing a sub-arc of P as 

Here, CD_P is  , 

Result 3 on Trivializing Number

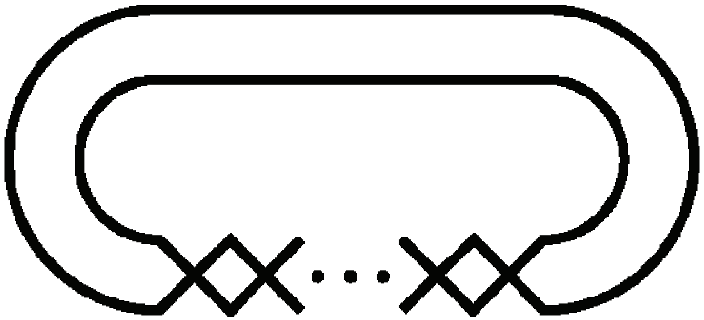
Theorem 1-6

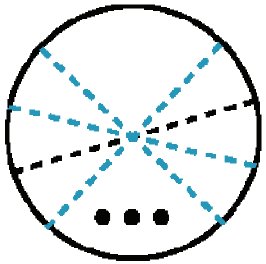
P : knot projection with pre-crossings

$\Rightarrow$ $\mathrm{tr}(P) \leqq \mathrm{p}(P)-1$

where $\mathrm{p}(P)$ is the number of the pre-crossings of P

The equality holds.

$\Leftrightarrow$ P is 

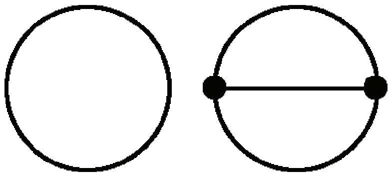
Here, CD_P is 

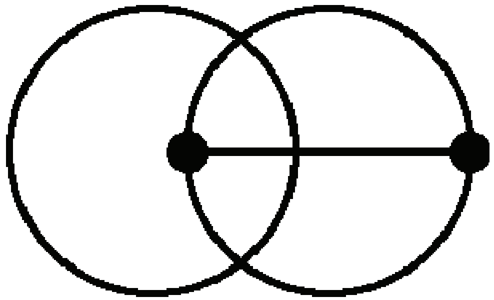
Trivializing Number of Spatial Graphs

Corollary 1-7

P : link projection ⇒ tr(P) is even

Remark 1-8

G = 

∃ P : projection of G s.t. tr(P) = 3 

On Pseudo Diagram for Virtual Knots

- A. Henrich etc. expand a notion of pseudo diagram for virtual knots.
  - “Classical and Virtual Pseudodiagram Theory and New Bounds on Unknotting Numbers and Genus”
- Then, they discuss relation between trivializing number and unknotting number (resp. genus) in the paper.

- Henrich-etc. and W.Johnson create a game on pseudo diagrams.
  - "A Midsummer Knot's Dream"
  - "The Knotting-Unknotting Game played on Sums of Rational Shadows", arXiv:1107.2635

Trivializing Number for Knots

K : knot

tr(K) :＝min{ tr(P) | A diagram D obtained from a projection P represents K }

We call tr(K) the trivializing number of K.

Note

tr(K) is always even by Theorem 1-1.

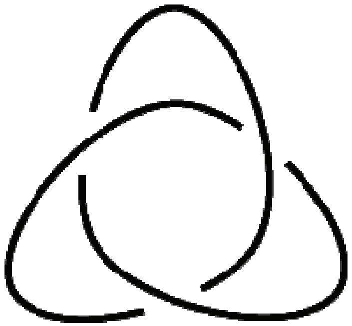
Crossing Number

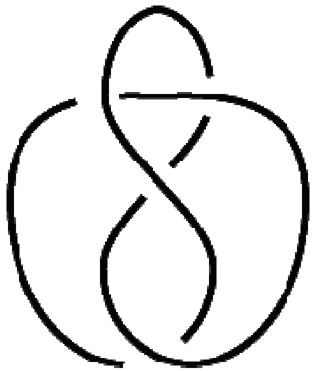
D : diagram of a knot K

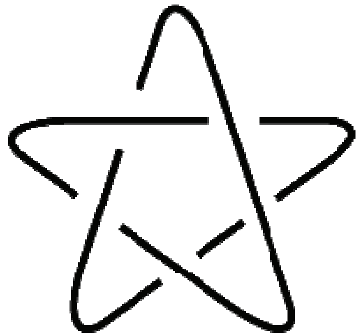
$c(D)$: the number of crossings of D

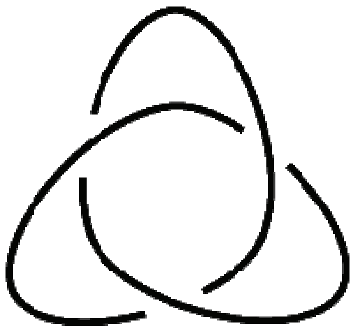
$c(K) := \min\{\ c(D) \mid D : \text{a diagram of } K\}$

We call $c(K)$ the crossing number of K



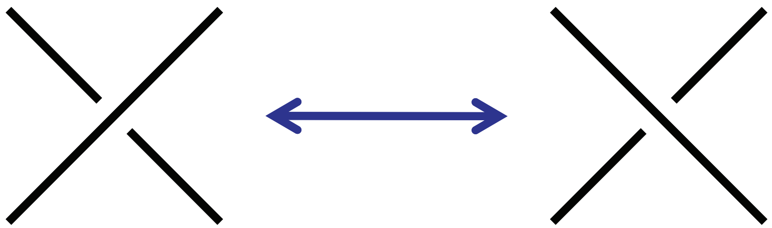




$c($  $) = 3$

Crossing Change and Trivial Knot Diagram

A crossing change is a local move on a diagram of a knot as following.

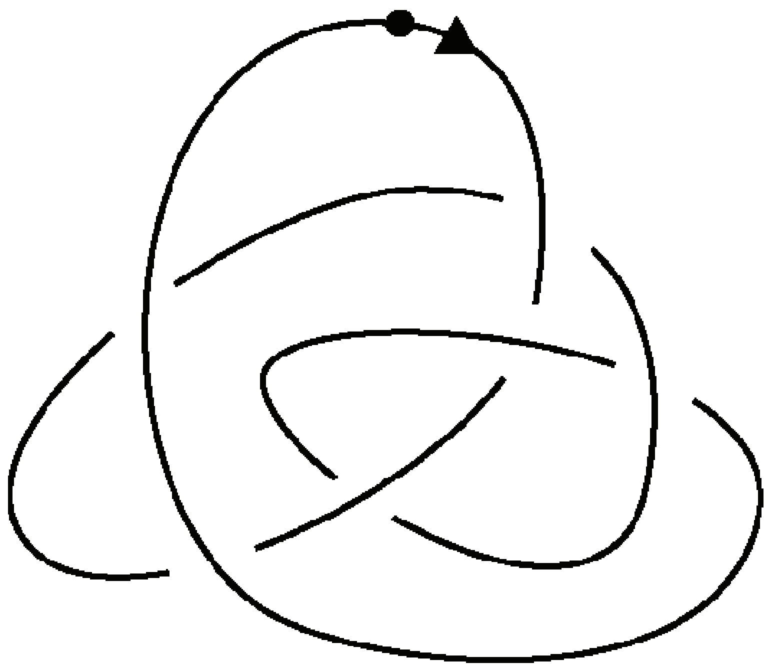


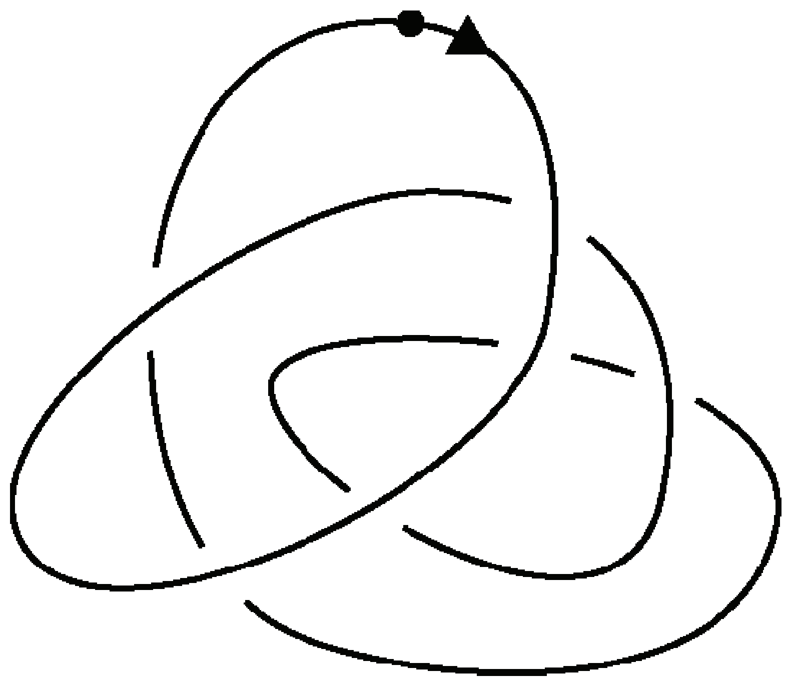
<u>Proposition</u>

$\forall D$: diagram of a knot

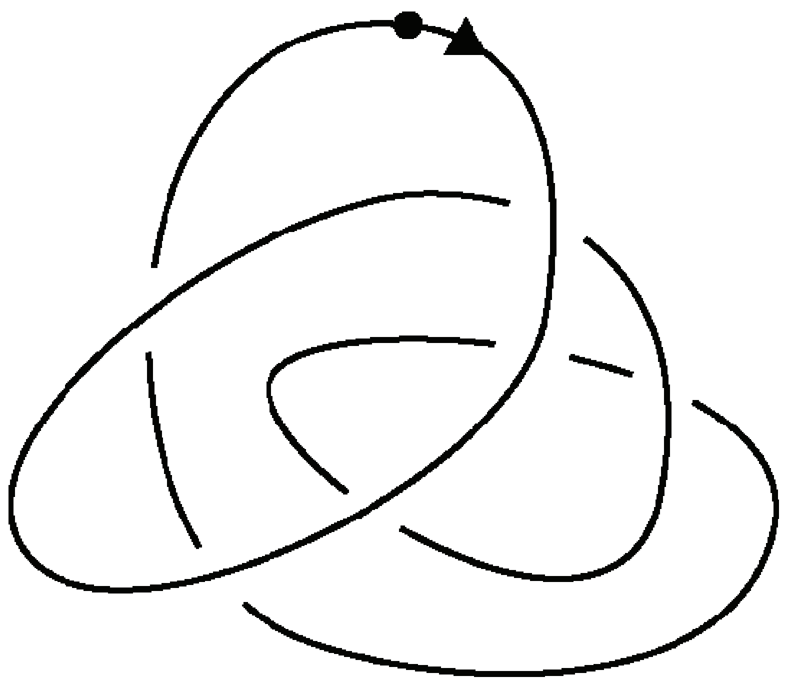
We get a diagram representing the trivial knot by crossing changes of some crossings of D.

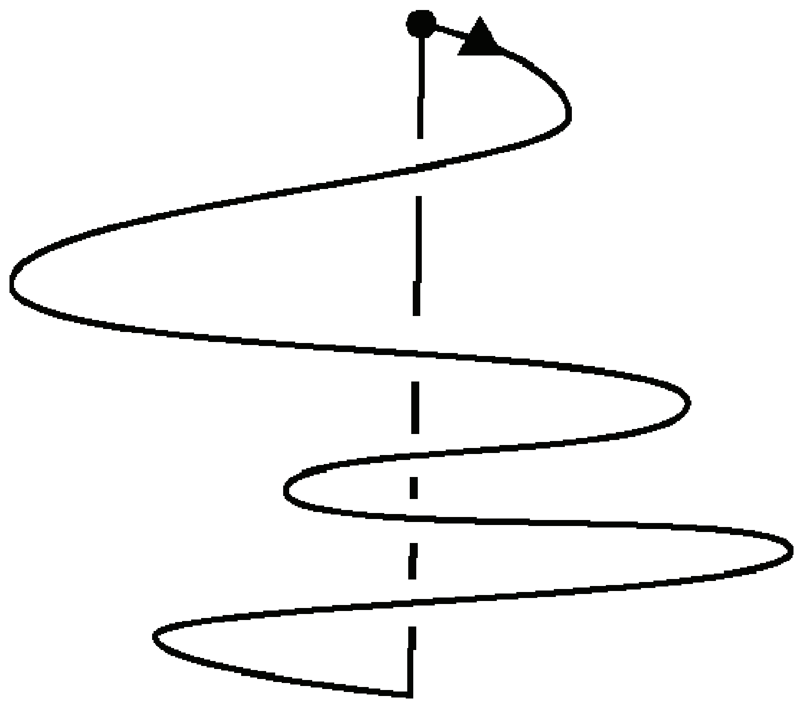
Proof





Proof





Unknotting Number of Knots

D : a diagram of a knot K

u(D):＝min{ n | changing some n crossings of D yields a diagram of the trivial knot}

u(K):＝min{ u(D) | D : a diagram of knot K}

We call u(D) the unknotting number of a diagram D, u(K) the unknotting number of a knot K.

u(K) ≦ u(D) holds.

Proposition

D : a diagram of knot K with a crossing

$$\Rightarrow u(K) \leq u(D) \leq \frac{c(D)-1}{2} \quad u(K) \leq \frac{c(K)-1}{2}$$

Trivializing Number and Unknotting Number

Proposition 2-1 [H '10, Henrich-etc. '11]

$$u(K) \leqq \frac{1}{2}\mathrm{tr}(K)$$

where u(K) is the unknotting of K

Proof. It follows from the definition of the trivializing number and a fact that a mirror diagram of a trivial knot is also trivial.

Trivializing Number and Genus

Theorem 2-2 [Henrich-etc.]

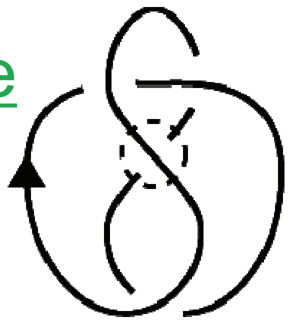
$$g(K) \leqq \frac{1}{2}\mathrm{tr}(K) \quad \text{where } g(K) \text{ is the genus of } K$$

Proposition 2-3

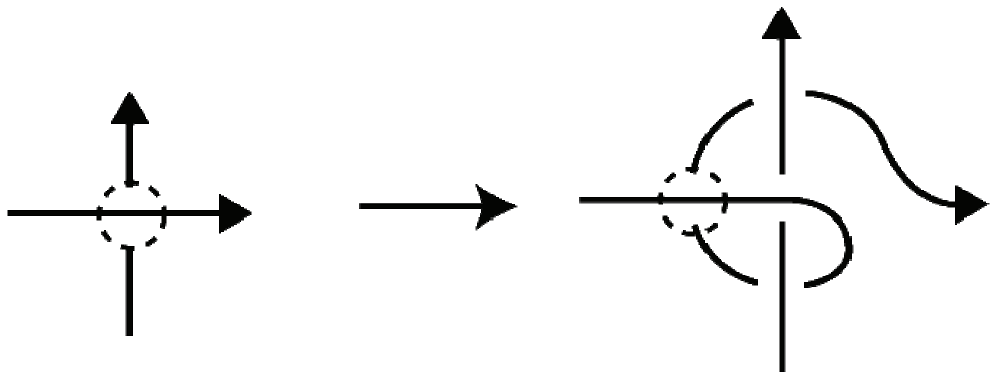
$\forall n$: non-negative number

$\exists K$: an alternating knot with $\frac{1}{2}\mathrm{tr}(K) - u(K) = n$

Example



D_0



Trivializing Number

<u>Theorem 2-4</u>

$\mathrm{tr}(K) = 2 \;\Leftrightarrow\; K$ is a twist knot

<u>Theorem 2-5</u>

K : nontrivial knot

$\Rightarrow\; \mathrm{tr}(K) \leqq \mathrm{c}(K) - 1$

where c(K) is the crossing number of K

The equality holds.

$\Leftrightarrow\; K$ is (2, p)-torus knot where p is some odd number

Trivializing Number of Positive Knots

Proposition 2-6

K : positive knot with up to 10 crossings

$\Rightarrow$ $\mathrm{tr}(K) = 2\mathrm{u}(K)$

Moreover,

P : the projection of some positive diagram of K,

$\mathrm{tr}(P) = \mathrm{tr}(K)$

Note [T. Nakamura '00]

There exist exactly 42 positive knots in up to 10 crossing knots.

Conjecture on Positive Knots

Conjecture $\forall K$: positive knot, tr(K)＝2u(K)
Moreover, $\forall D$: positive diagram of K, tr(D)＝tr(K)

Question [Stoimenow '03]
Does every positive knot realize its unknotting number in a positive diagram?

Theorem 2-7
K : positive braid knot ⇒ tr(K)＝2u(K)
Moreover, D : positive braid diagram of K
⇒ tr(D)＝2u(K)

Positive Diagram and Four Genus

Theorem 2-8 [T.Nakamura '00, Rasmussen '04]

D : positive diagram

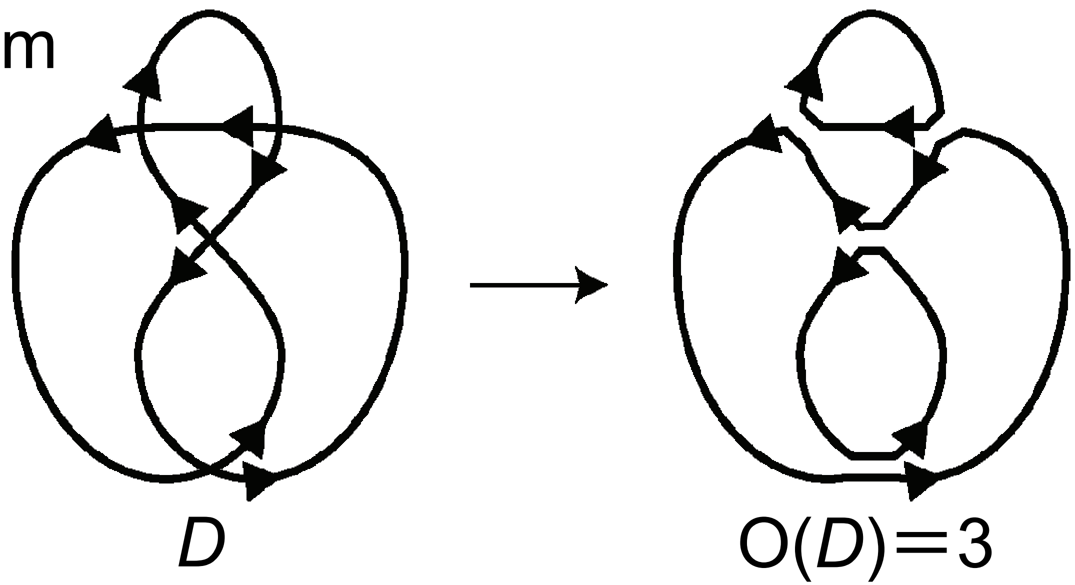
K : the knot represented by D

$$2g_4(K)=2g(K)=c(D)-O(D)+1$$

where $c(D)$ is the number of the crossings

and $O(D)$ is the number of the Seifert circles

and $g_4(K)$ is the minimum genus of a surface locally flatly embedded in the 4-ball with boundary K



Proposition 2-9

$u(K) \geqq g_4(K)$

Proof of Theorem 2.7

<u>Sketch Proof of Theorem 2-7.</u>

D : positive braid diagram of K

P : the projection of D

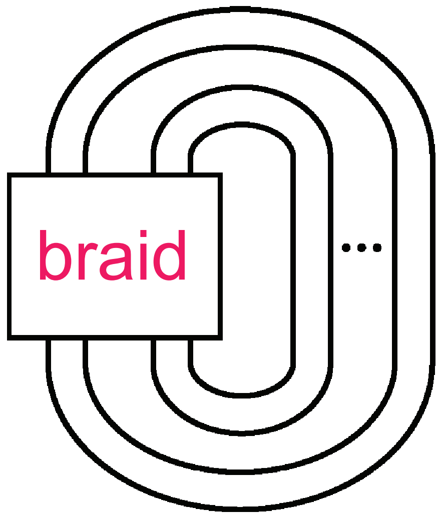
By Propositions 2-1 and 2-9 and Theorem 2-8,

$$\mathrm{tr}(P) \geqq \mathrm{tr}(K) \geqq 2\mathrm{u}(K) \geqq 2\mathrm{g}_4(K) = c(D) - O(D) + 1$$

On the other hand,

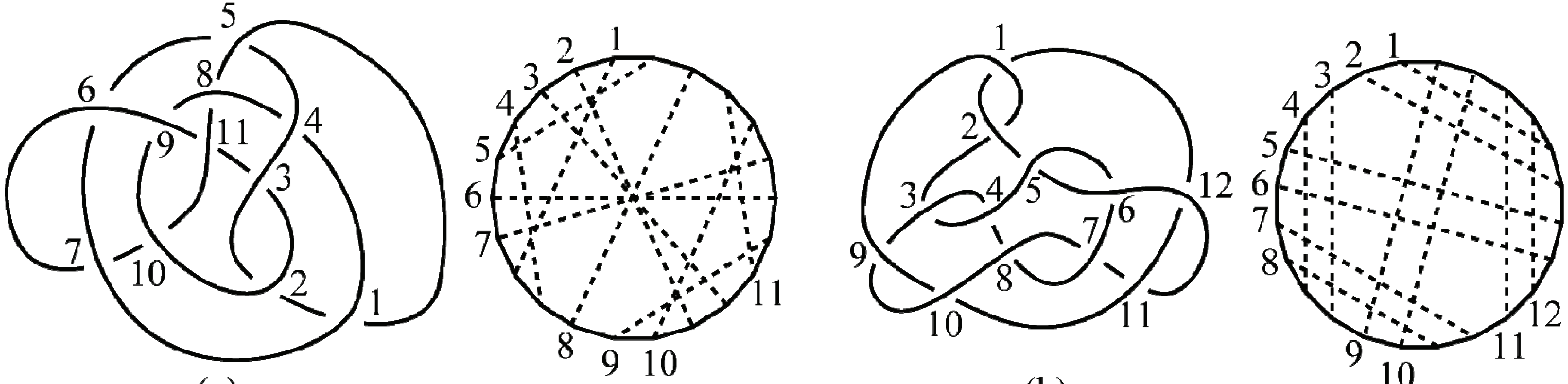
$\mathrm{tr}(P) = c(D) - O(D) + 1.$

Therefore, $\mathrm{tr}(K) = 2\mathrm{u}(K)$.■



Minimal Diagram and Trivializing Number

Proposition 2-10 The knot 11_{550} does not have its trivializing number in minimal crossing diagrams. The positive 12 crossing diagram (b) realizes the trivializing number of 11_{550}.



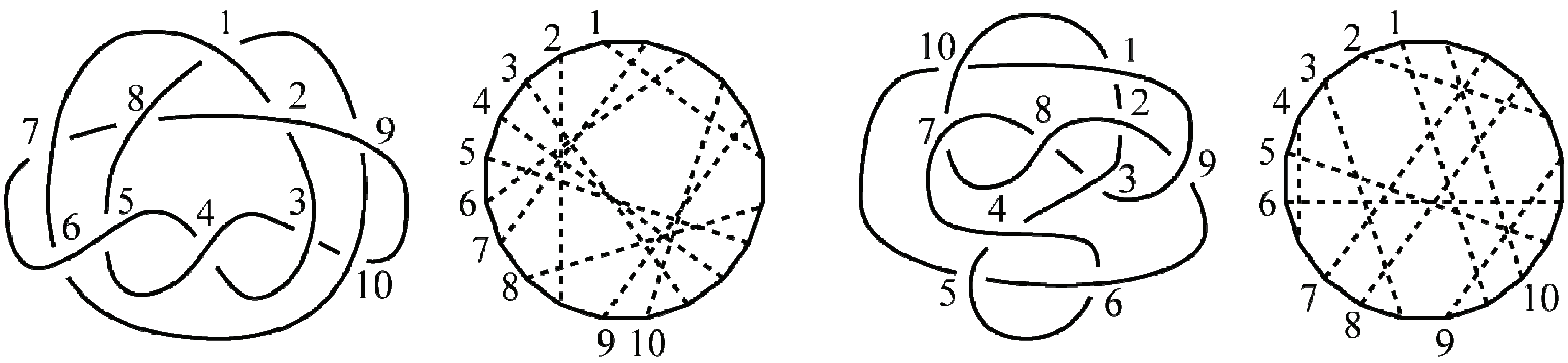
(a) (b)

Note [Stoimenow '02] 11_{550} has only one 11 crossing diagram (a) which is not positive but has a positive 12 crossing diagram (b).

Minimal Diagram and Trivializing Number

There exists a knot whose minimal crossing diagrams have different trivializing number.

For example, Perko's pair which represent 10_{161} have different trivializing number.



(a) (b)

<u>Remark</u> D, D' : alternating diagram of K

$\Rightarrow$ $\mathrm{tr}(D) = \mathrm{tr}(D')$

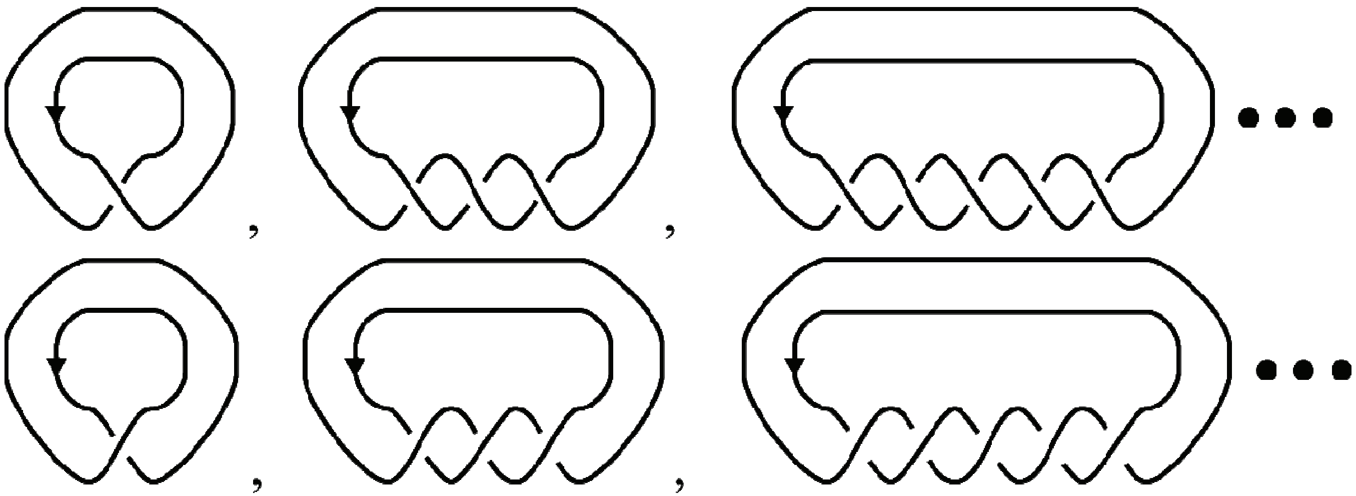
Application to Unknotting Number

Theorem [Taniyama '09]

D : a diagram of a knot *K*

$$u(D) = \frac{c(D) - 1}{2}$$

⇔ *D* is a (2, *p*)-torus knot diagram for some odd integer *p*.



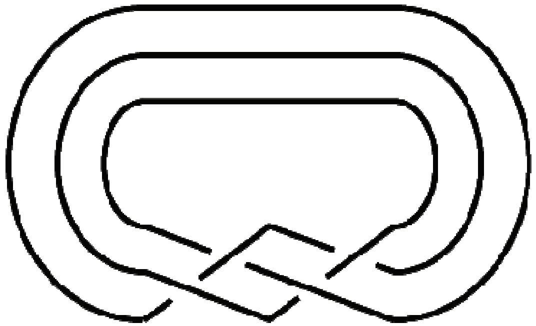
Application to Unknotting Number

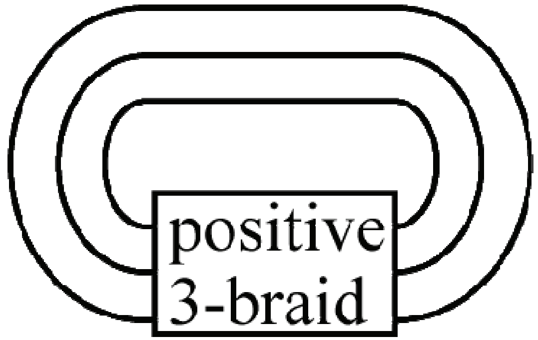
Theorem [Abe-H-Higa '10]

D : a diagram of a knot *K*

$$u(D) = \frac{c(D) - 2}{2}$$

⇔ *D* is one of the figure-eight knot diagram, the positive 3-braid knot diagrams, the mirror diagrams of them and the connected sum of a (2,*r*)-torus knot diagram and a (2,*s*)-torus knot diagram for some odd integers *r*, *s* ≠ ±1.





Application to a Partial Order of Knots

Proposition [Taniyama '89] L_1, L_2 : μ-component links

$L_1 \leqq L_2$

$\Rightarrow$ c(L_1) $\leqq$ c(L_2), br(L_1) $\leqq$ br(L_2), b(L_1) $\leqq$ b(L_2)

where c(L) : the minimum number of crossings of L

br(L) : the bridge index of L

b(L) : the braid index of L

Proposition [H '10]

K_1, K_2 : knots

$K_1 \leqq K_2 \Rightarrow$ tr(K_1) $\leqq$ tr(K_2)

ご清聴ありがとうございました