Trivializing number of knots

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Contents

Definition of projection

K : an oriented knot in $\mathbf{R}^3$

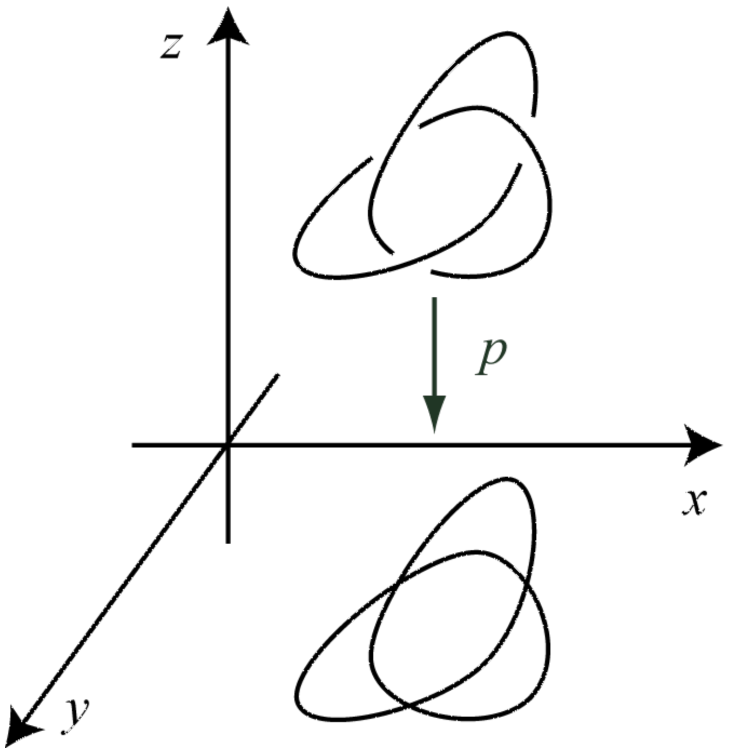
$p : \mathbf{R}^3 \to \mathbf{R}^2$: natural projection

p is a projection of a knot K

$\Leftrightarrow$ multiple points of $p|_K$ are
only finitely many transversal
double points.

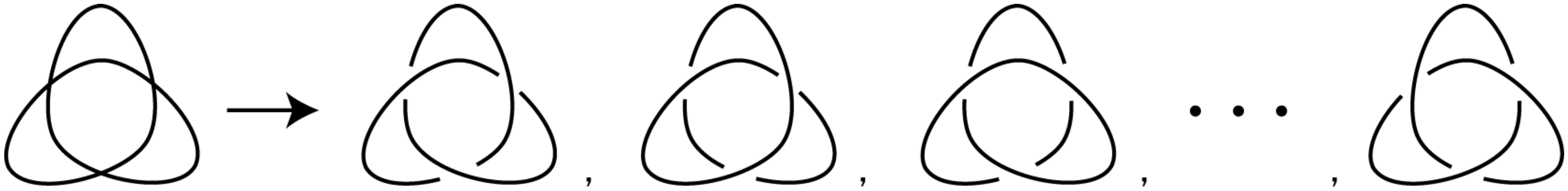
We call $p(K)$ a (knot) projection

and denote it by $P = p(K)$.



Motivation on pseudo diagram

- Which double points of a projection and which over/under informations at them should we know in order to determine that the original knot is trivial or knotted?



- We introduced a notion of the pseudo diagram in [H, 2010].

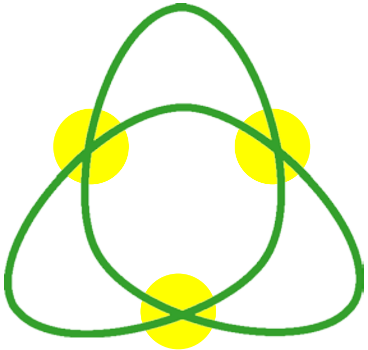
Definition of diagram

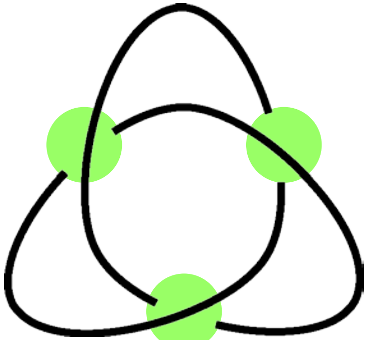
A diagram D is a projection P with over/under information

at every double point.

Then we say D is obtained from P and P is the projection of D.

A diagram uniquely represents a knot up to equivalence.





Then a double point with (resp. without) over/under information is called a crossing (resp. a pre-crossing).

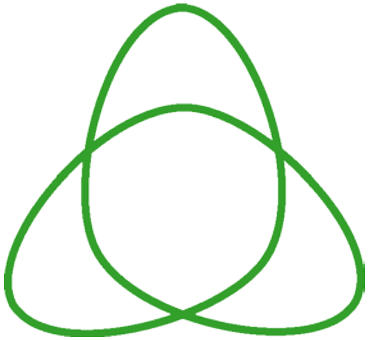
Definition of pseudo diagram

A pseudo diagram *Q* is a projection *P* with over/under information at some pre-crossings.

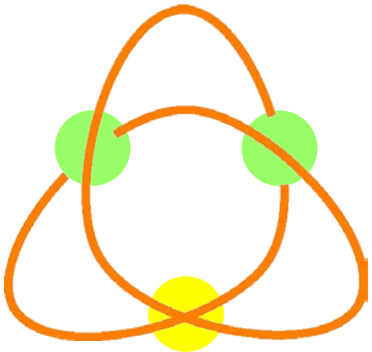
Thus, a pseudo diagram *Q* has crossings and pre-crossings.

Here, *Q* possibly has no crossings or no pre-crossings.

Namely, *Q* is possibly a projection or a diagram.



P



Q

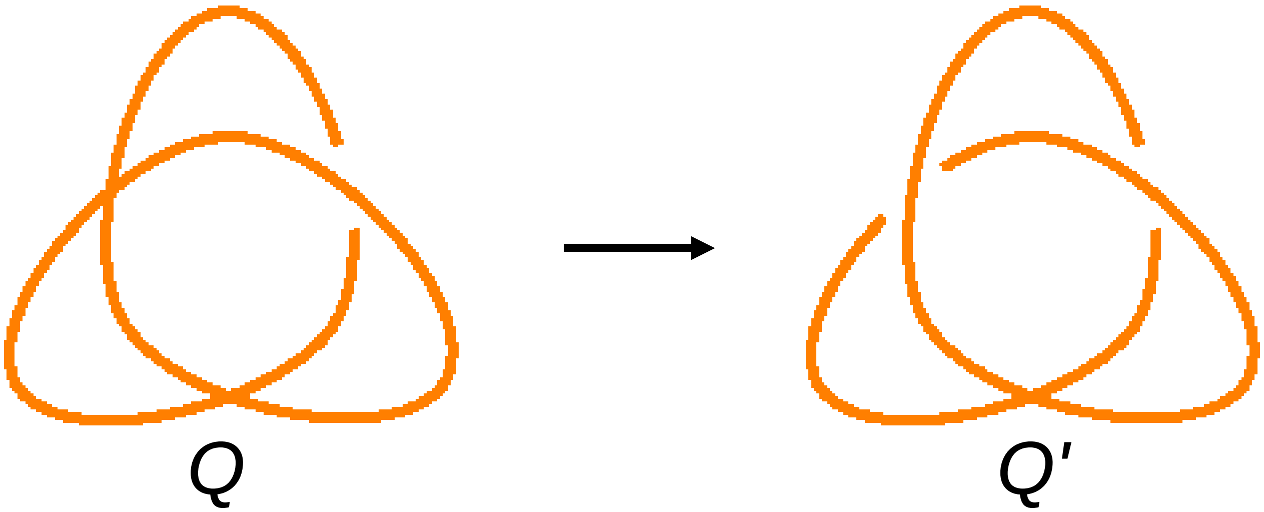
Relation between pseudo diagrams

Q, Q' : pseudo diagrams of a projection

A pseudo diagram Q' is obtained from a pseudo diagram Q.

⇔ Each crossing of Q has the same over/under information as Q'.

Ex.

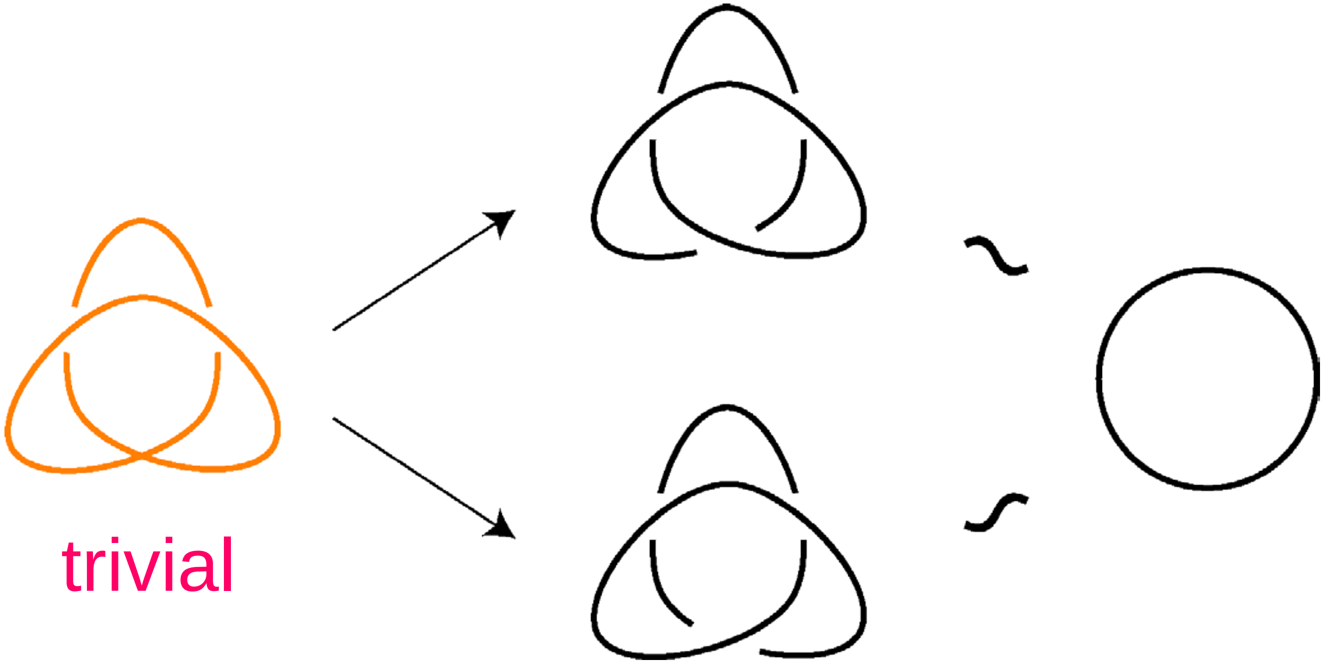


Trivial of pseudo diagrams

A pseudo diagram *Q* is trivial.

⇔ Any diagram obtained from *Q* represents a trivial knot.

Ex.



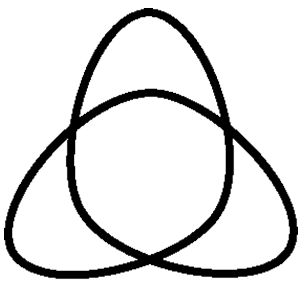
Trivializing number

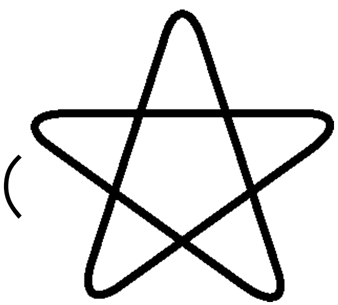
tr(*P*):＝min{ c(*Q*) | *Q* : trivial pseudo diagram obtained from *P* }

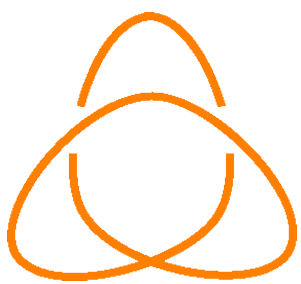
where c(*Q*) is the number of the crossings of *Q*.

We call tr(*P*) the trivializing number of *P*.

Ex.

tr（  ）＝2

tr（  ）＝4

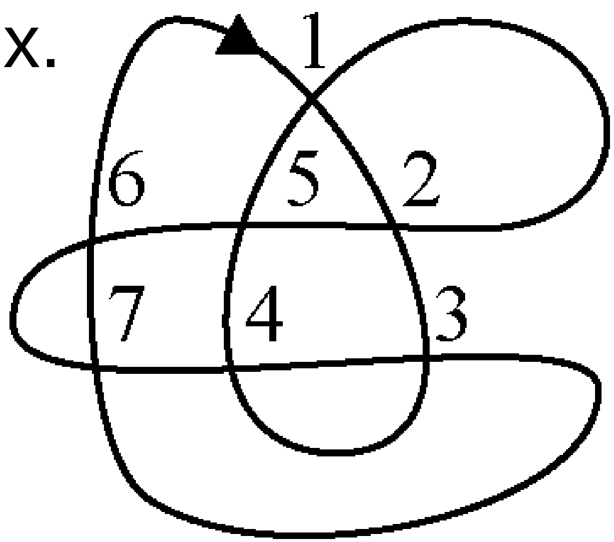


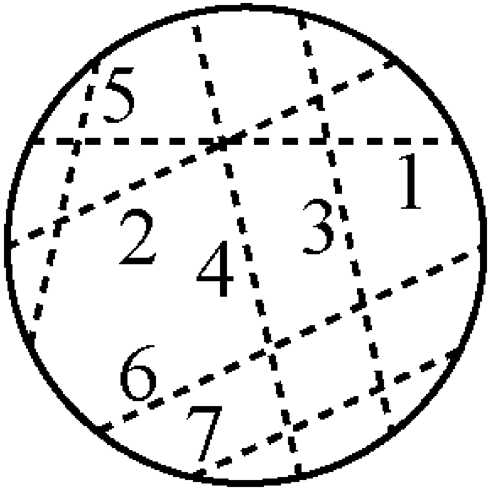
Chord diagram of a projection

P : a knot projection with *n* pre-crossings

A chord diagram of *P* is a circle with *n* chords marked on it by dashed line segment, where the pre-image of each pre-crossing is connected by a chord.

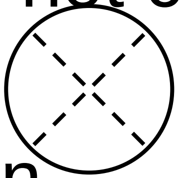
Ex.





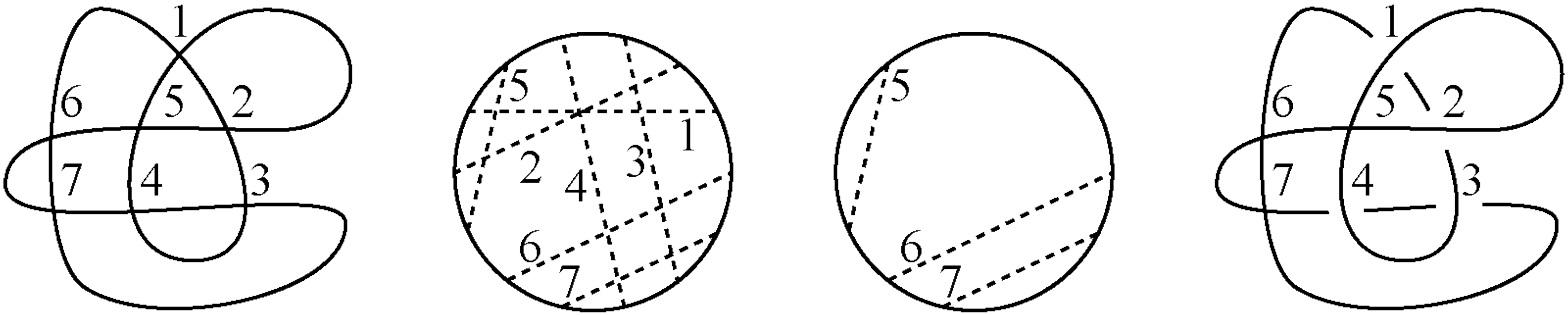
Trivializing number and chord diagram

Theorem 1-1 [H, 2010] P : a knot projection

$\mathrm{tr}(P)$ = min{ n | Deleting some n chords from CD_P yields a chord diagram which does not contain a sub-chord diagram as  }

and $\mathrm{tr}(P)$ is even.

Ex. $\mathrm{tr}(P) = 4$

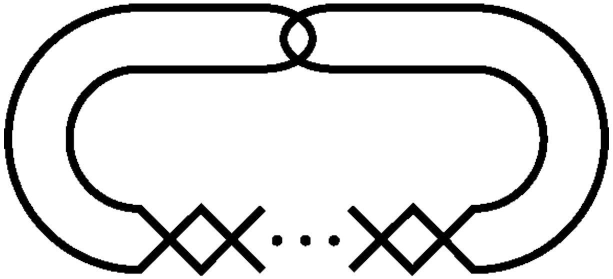


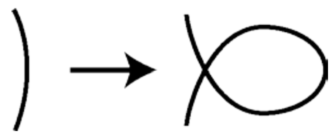
Result on trivializing number

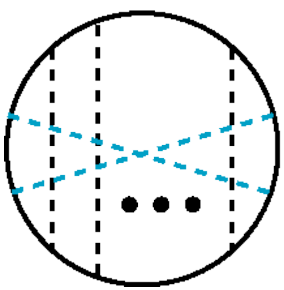
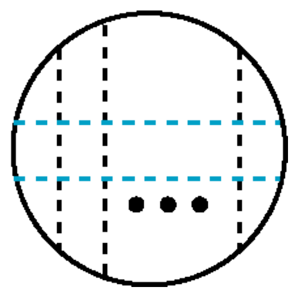
Theorem 1-2 [H, 2010]

P : knot projection

tr(P)＝2

⇔ P is obtained from 

by a series of replacing a sub-arc of P as 

Here, CD_P is , 

Result on trivializing number

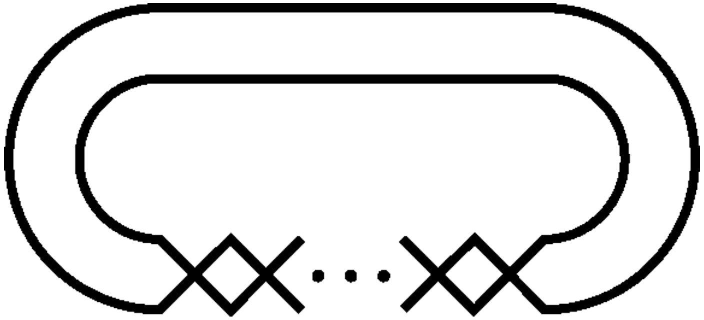
Theorem 1-3 [H, 2010]

P : knot projection with pre-crossings

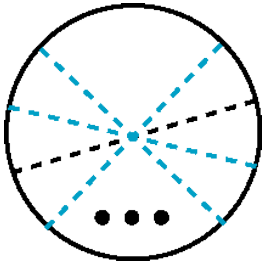
⇒ $\mathrm{tr}(P) \leqq \mathrm{p}(P) - 1$

where $\mathrm{p}(P)$ is the number of the pre-crossings of P

The equality holds ⇔ P is



Here, CD_P is



Trivializing number of knots

$\mathrm{tr}(D) :=$ $\mathrm{tr}(P)$ where P is the projection of D

$\mathrm{tr}(K) := \min\{\, \mathrm{tr}(D) \mid \text{A diagram } D \text{ represents } K \,\}$

We call $\mathrm{tr}(K)$ the trivializing number of K.

Note $\mathrm{tr}(K)$ is always even by Theorem 1-1.

- A. Henrich etc. expand a notion of pseudo diagram for virtual knots.
  - arXiv:0908.1981v2
- Then, they discuss relation between trivializing number and unknotting number (resp. genus) in the paper.

Trivializing number and unknotting number

Proposition 2-1 [H, Henrich-etc.]

$$u(K) \leqq \frac{1}{2}\mathrm{tr}(K)$$

where u(K) is the unknotting of K

Proof. It follows from the definition of the trivializing number and a fact that a mirror diagram of a trivial knot is also trivial.

Theorem 2-2 [Henrich-etc.]

$$g(K) \leqq \frac{1}{2}\mathrm{tr}(K)$$

where g(K) is the genus of K

Results

Theorem 2-3

$\mathrm{tr}(K) = 2 \iff K$ is a twist knot

Proof. It follows from Theorem 1-2.

Theorem 2-4

K : nontrivial knot $\Rightarrow 2 \leqq \mathrm{tr}(K) \leqq c(K) - 1$

where $c(K)$ is the crossing number of K

$\mathrm{tr}(K) = c(K) - 1 \iff K$ is a $(2, p)$-torus knot

Proof. It follows from Theorem 1-3.

Trivializing number of positive knots

Proposition 2-5

K : positive knot with up to 10 crossings

$\Rightarrow$ $\mathrm{tr}(K) = 2\mathrm{u}(K)$

Moreover,

P : the projection of some positive diagram of K,

$\mathrm{tr}(P) = \mathrm{tr}(K)$

Note [T. Nakamura '00]

There exist exactly 42 positive knots in up to 10 crossing knots.

Conjecture on positive knots

Conjecture $\forall K$: positive knot, $\mathrm{tr}(K) = 2\mathrm{u}(K)$
Moreover,

$\forall D$: positive diagram of K, $\mathrm{tr}(D) = \mathrm{tr}(K)$

Question [Stoimenow '03]
Does every positive knot realize its unknotting number in a positive diagram?

Theorem 2-6

K : positive braid knot ⇒ $\mathrm{tr}(K) = 2\mathrm{u}(K)$

Moreover, D : positive braid diagram of K

⇒ $\mathrm{tr}(D) = 2\mathrm{u}(K)$

Positive diagram and four genus

Theorem 2-7 [T.Nakamura '00, Rasmussen '04]

K : positive knot, *D* : positive diagram of *K*

$$2g_4(K) = 2g(K) = c(D) - O(D) + 1$$

where $c(D)$ is the number of the crossings

and $O(D)$ is the number of the Seifert circles

and $g_4(K)$ is the minimum genus of a surface locally flatly embedded in the 4-ball with boundary *K*

Note $s(K) = c(D) - O(D) + 1$, $s(K)$ is the Rasmussen invariant for a positive knot *K* and a positive diagram *D* of *K*.

Proposition 2-8 $u(K) \geqq g_4(K)$

Proof of Theorem 2.6

Sketch Proof of Theorem 2-6.

D : positive braid diagram of K

P : the projection of D

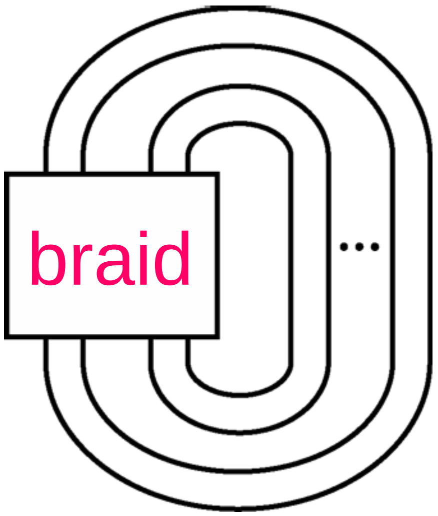
By Propositions 2-1 and 2-8 and Theorem 2-7,

$$\mathrm{tr}(P) \geqq \mathrm{tr}(K) \geqq 2\mathrm{u}(K) \geqq 2\mathrm{g}_4(K) = c(D) - O(D) + 1$$

On the other hand,

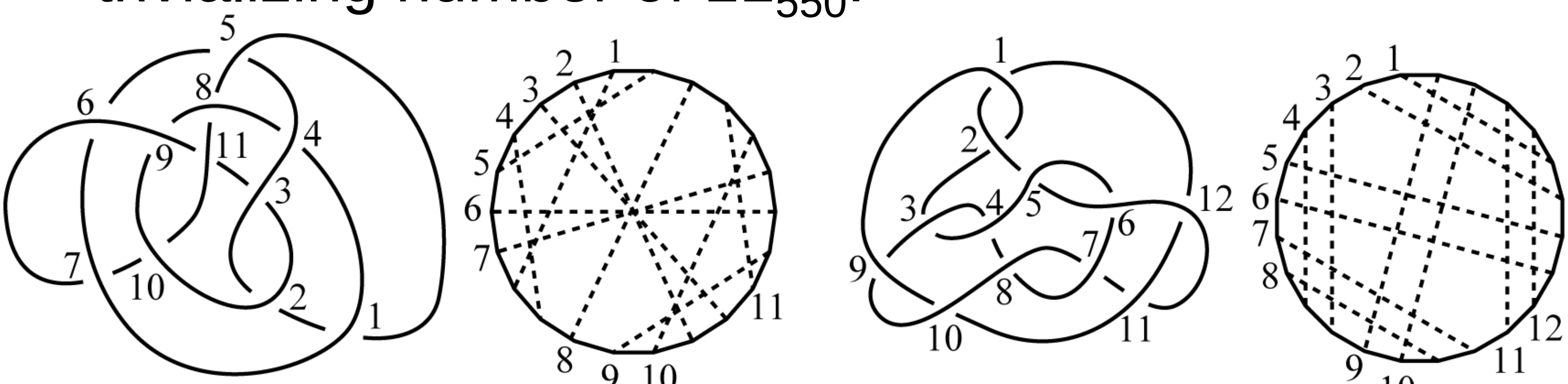
$\mathrm{tr}(P) = c(D) - O(D) + 1.$

Therefore, $\mathrm{tr}(K) = 2\mathrm{u}(K)$. ■



Mimimal diagram and trivializing number

Proposition 2-9 The knot 11_{550} does not have its trivializing number in minimal crossing diagrams. The positive 12 crossing diagram (b) realizes the trivializing number of 11_{550}.



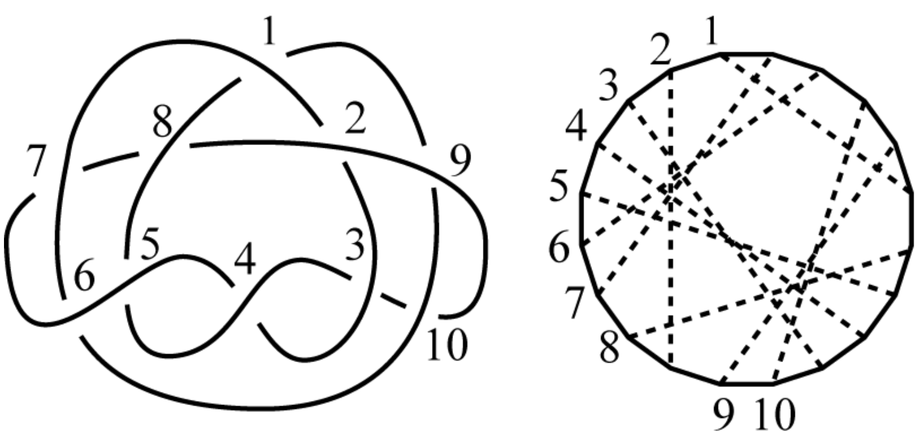
(a) (b)

Note [Stoimenow '02] 11_{550} has only one 11 crossing diagram (a) which is not positive but has a positive 12 crossing diagram (b).

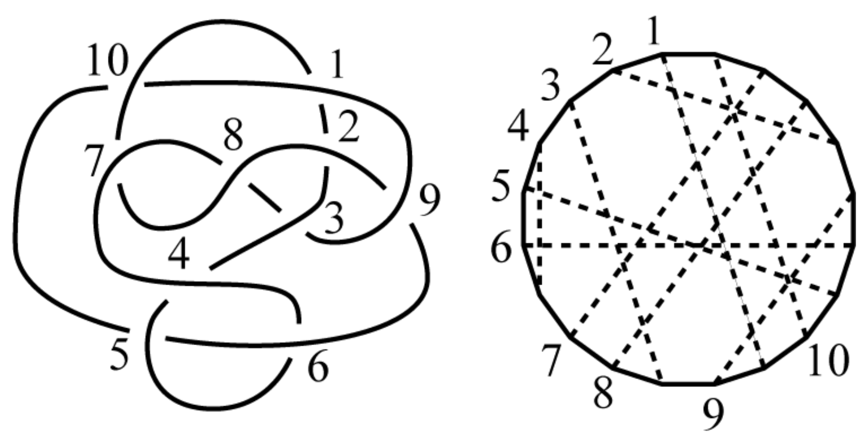
Mimimal diagram and trivializing number

There exists a knot whose minimal crossing diagrams have different trivializing number.

For example, Perko's pair which represent 10_{161} have different trivializing number.



(a)



(b)

<u>Remark</u> D, D' : alternating diagram of K

$\Rightarrow$ $\mathrm{tr}(D) = \mathrm{tr}(D')$

ご清聴ありがとうございました