On an inequality between unknotting number and crossing number of links

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Definitions & Notations

L : oriented link in S^3, D : diagram of L on S^2

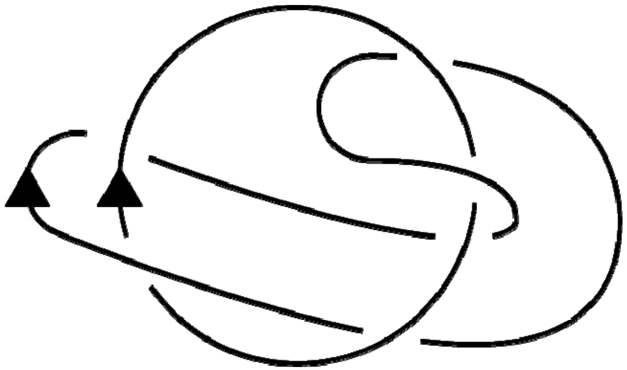
$u(D)$:= min{n | changing some n crossings of D yields a trivial link diagram}

$u(L)$:= min{$u(D)$ | D : diagram of L}

ex. L : Whitehead link

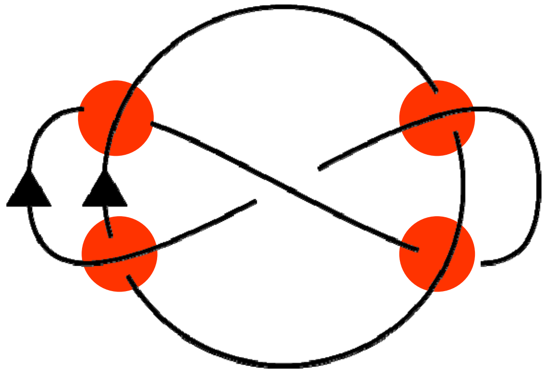
: mutual crossing

D :



$u(D) = 2$

D' :



$u(D') = 1$

$u(L) = 1$

Folklore

$c(D)$: the number of crossings in D

$c(L) := \min\{c(D) \mid D : \text{diagram of } L\}$

D is a minimal diagram of $L \Leftrightarrow c(D) = c(L)$

Proposition 1

$$u(L) \le u(D) \le \frac{c(D)}{2}, \quad u(L) \le \frac{c(L)}{2}$$

if D is a diagram of a link L

$$u(K) \le u(D) \le \frac{c(D)-1}{2}, \quad u(K) \le \frac{c(K)-1}{2}$$

if D is a diagram of a knot K

Known Results

Theorem 1 [Taniyama, 2008]

$L = \gamma_1 \cup \gamma_2 \cup \cdots \cup \gamma_\mu$: μ-component link

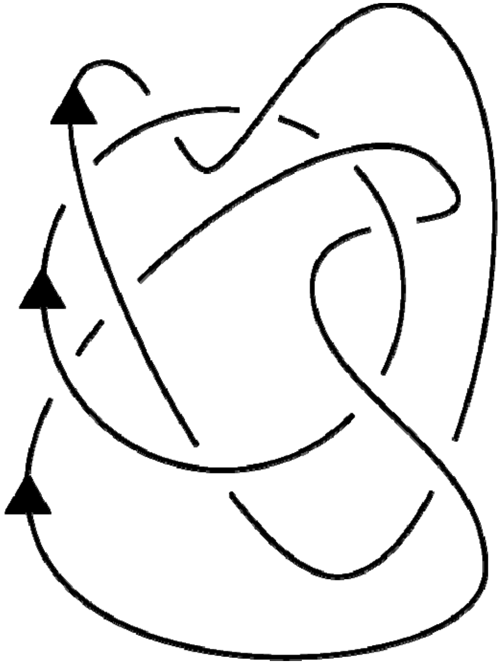
D : diagram of L with $u(D) = \dfrac{c(D)}{2}$

$\Rightarrow$ each γ_i is a simple closed curve and subdiagram $\gamma_i \cup \gamma_j$ $(1 \leqq i < j \leqq \mu)$ is an alternating diagram or a diagram without crossings.

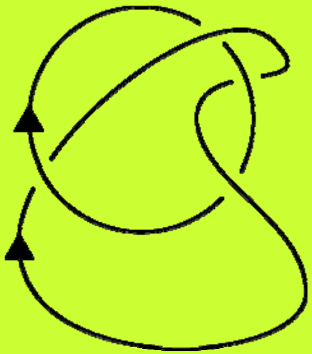
Note that the diagram $\gamma_i \cup \gamma_j$ $(1 \leqq i < j \leqq \mu)$ is also a positive diagram or a negative diagram.

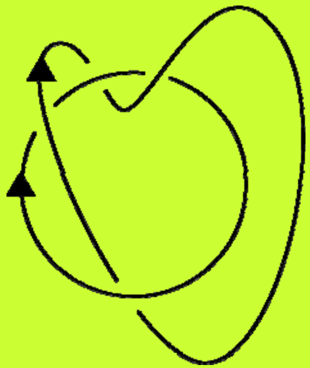
Known Results

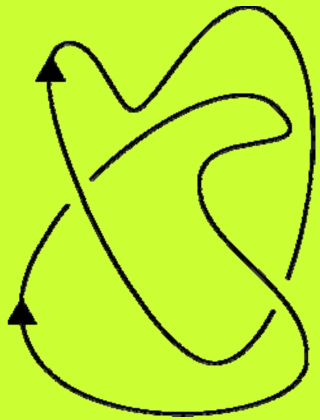
ex.



subdiagrams :







Known Results

Theorem 2 [Taniyama, 2008]

L : link with $u(L) = \frac{c(L)}{2}$

$\Rightarrow$ L has a diagram D with $u(D) = \frac{c(D)}{2}$

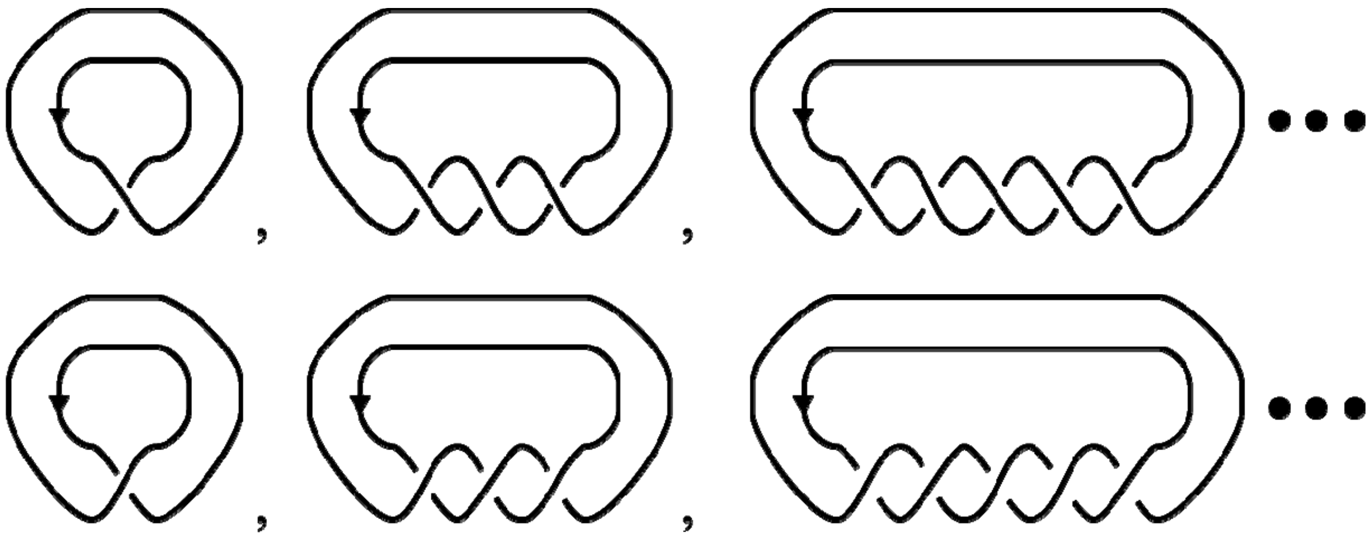
In addition, it holds that for any minimal diagram D of L, $u(D) = \frac{c(D)}{2}$.

Known Results

Theorem 3 [Taniyama, 2008]

D : diagram of a knot with $u(D) = \dfrac{c(D)-1}{2}$

$\Rightarrow$ D is one of the diagrams as



Note that each diagram is an alternating positive diagram or an alternating negative diagram.

Known Results

Theorem 4 [Taniyama, 2008]

K : knot with $u(K) = \dfrac{c(K)-1}{2}$

$\Rightarrow K$ has a diagram D with $u(D) = \dfrac{c(D)-1}{2}$

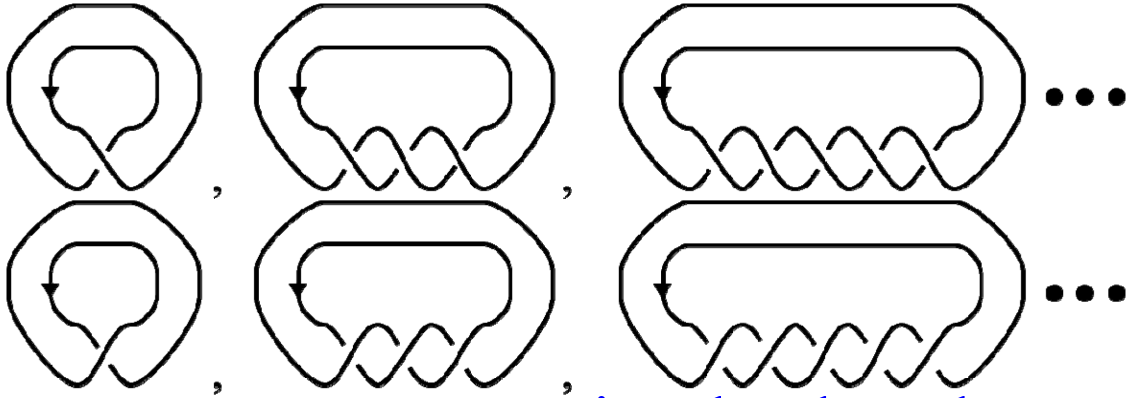
Main Result

Main Theorem

$L = \gamma_1 \cup \gamma_2 \cup \cdots \cup \gamma_\mu$: μ-component link, D : diagram of L

$$u(D) = \frac{c(D) - 1}{2}$$

$\Leftrightarrow$ just one component of D is one of the diagrams as

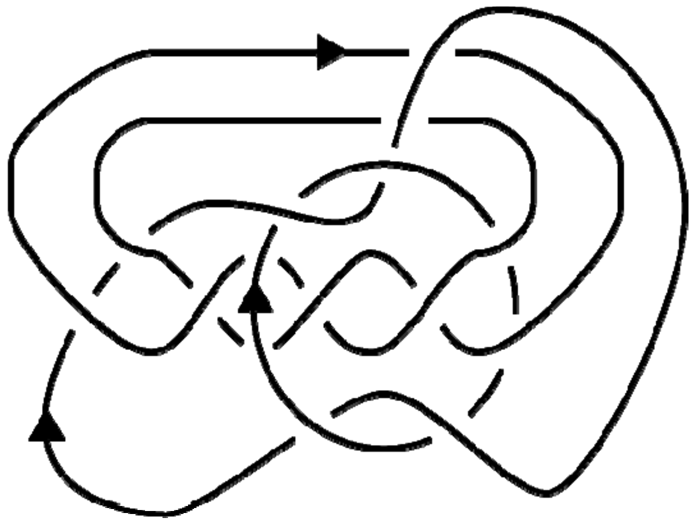


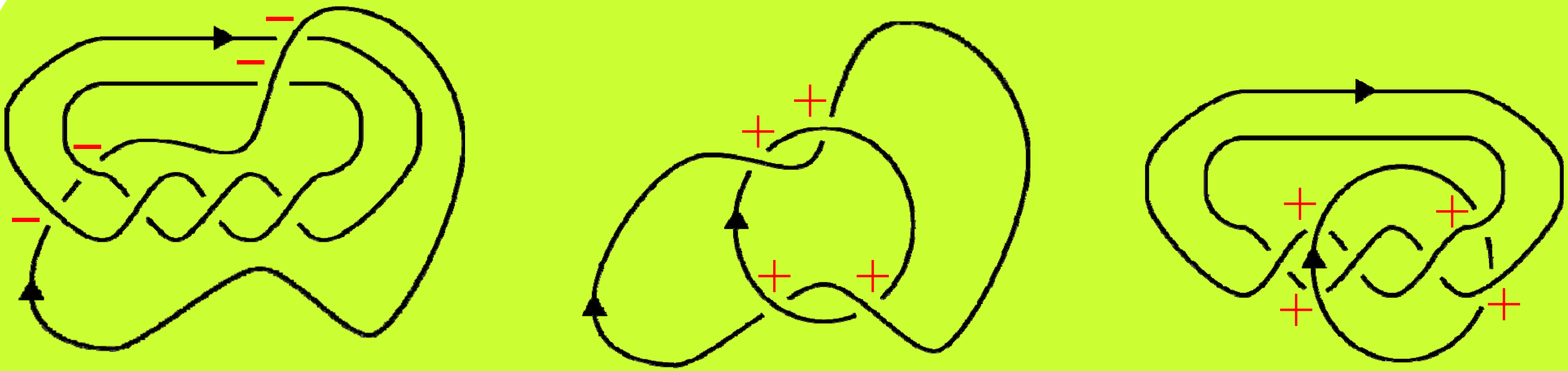
, the other components are simple closed curves

, and the mutual crossings of subdiagram $\gamma_i \cup \gamma_j$ $(1 \leqq i < j \leqq \mu)$ are all positive, all negative, or nothing.

Main Result

ex.





the subdiagrams

Characterization

Corollary 1

L : link with $u(L)=\dfrac{c(L)-1}{2}$

$\Rightarrow L$ has a diagram D with $u(D)=\dfrac{c(D)-1}{2}$

Proof of Corollary 1

D : a minimal diagram of L, that is, $c(D) = c(L)$

$$\frac{c(D)}{2} \geq u(D) \geq u(L) = \frac{c(L)-1}{2} = \frac{c(D)-1}{2}$$

Here, $c(L)$ is odd, and so is $c(D)$.

We never admit $u(D) = \dfrac{c(D)}{2}$.

Therefore, $u(D) = \dfrac{c(D)-1}{2}$. ■

On the following slides

First we introduce corollaries on diagrams D with $u(D) \geq \dfrac{c(D)-1}{2}$ by Theorems 1 and 2 and Main Theorem.

Last we introduce a corollary and problems on the relations between unknotting number and minimal diagrams from the corollaries obtained above.

Corollary 2

Corollary 2

D : diagram with $u(D) \geq \dfrac{c(D)-1}{2}$

$\Rightarrow D$ represents the link L with $u(L) = u(D)$

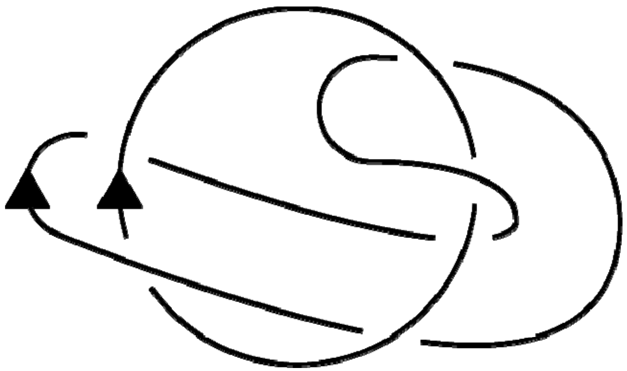
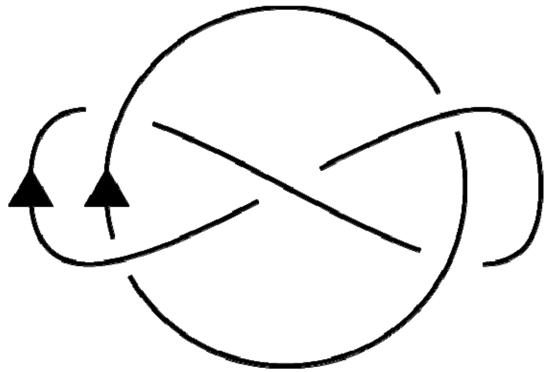
Proof. It follows from the signature and linking number of the links presented by D in Theorems 1 and 2 and Main Theorem. ■

Remark

$\exists D$: diagram with $u(D) = \dfrac{c(D)-2}{2}$

which represents the link L with $u(L) \neq u(D)$

ex.

D :  $\sim$ 

$u(D) = 2 = \dfrac{6-2}{2}$ $\qquad$ $u(L) = 1$

Corollary 3

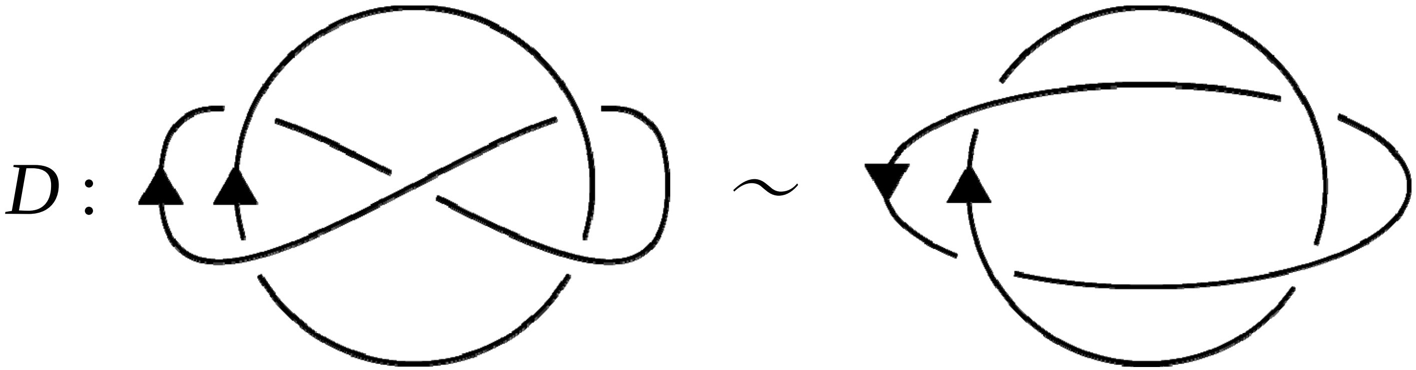
Corollary 3

D : diagram with $u(D) \geq \dfrac{c(D)-1}{2}$

L : the link represented by D

$\Rightarrow c(D)-1 \leqq c(L) \leqq c(D)$

ex. $u(D) = \dfrac{c(D)-1}{2}$ and $c(D)-1 = c(L)$



Corollary 4

Corollary 4

D : diagram with $u(D) \geq \dfrac{c(D)-1}{2}$

L : the link represented by D

$$\Rightarrow \quad u(L) = \frac{c(L)}{2} \quad \text{or} \quad u(L) = \frac{c(L)-1}{2}$$

Relations between unknotting number and minimal diagrams

$\exists K$: knot which has no minimal diagrams D with

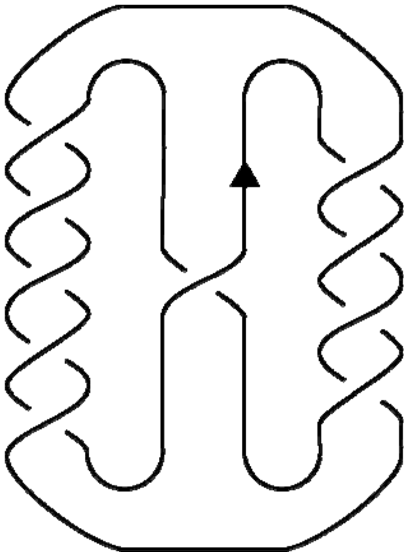
$$u(D) = u(K)$$

ex. [Nakanishi, 1983] and [Bleiler, 1984]

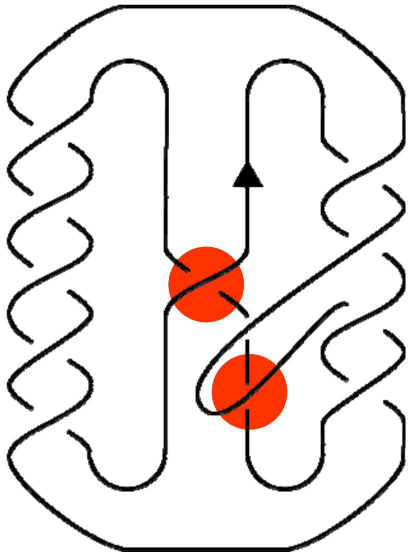
K : P(5,1,4) (=10_8)

$u(K) = 2 = u(D')$

$u(D) = 3$



D : the minimal diagram



D'

Relations between unknotting number and minimal diagrams

$\exists L$: link which has no minimal diagrams D with

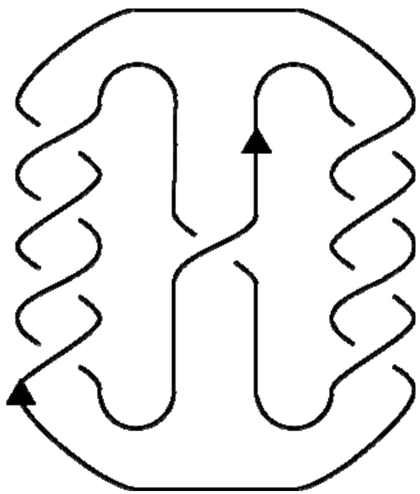
$$u(D) = u(L)$$

ex.

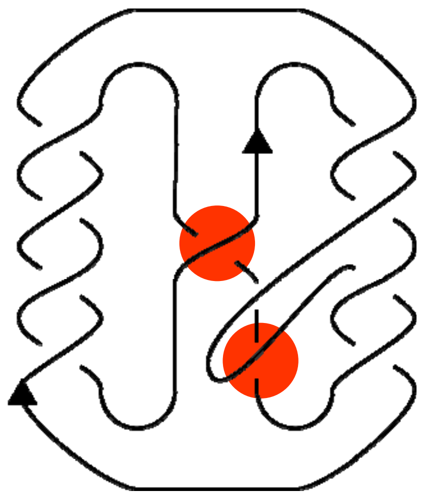
L : P(4,1,4)

$u(K) = 2 = u(D')$

$u(D) = 3$



D : the minimal diagram



D'

Corollary 5

Corollary 5

L : link with $u(L) \geq \frac{c(L)-2}{2}$

D : a minimal diagram of L, that is, $c(D)=c(L)$

$\Rightarrow u(D)=u(L)$

Proof of Corollary 5

If $u(L) \geq \frac{c(L)-1}{2}$, it follows from Theorem 2 and the proof of Corollary 1.

If $u(L) = \frac{c(L)-2}{2}$, by Corollary 4, $u(D) \leq \frac{c(D)-2}{2}$.

Therefore, $\frac{c(L)-2}{2} = u(L) \leq u(D) \leq \frac{c(D)-2}{2} = \frac{c(L)-2}{2}$. ■

Problem

Problem 1

Find minimum number n such that

$$\exists K : \text{knot with } u(K) = \frac{c(K) - n}{2}$$

which has no minimal diagrams D with $u(D) = u(K)$

We see from Corollary 5 and a P(5,1,4) knot K

with $u(K) = 2 = \dfrac{10-6}{2}$ that $3 \leqq n \leqq 6$.

Problem

Problem 2

Find minimum number n such that

$$\exists L : \text{link with } \; u(L) \geq \frac{c(L)-n}{2}$$

which has no minimal diagrams D with $u(D) = u(L)$

We see from Corollary 5 and a P(4,1,4) link L

with $u(L) = 2 = \dfrac{9-5}{2}$ that $3 \leqq n \leqq 5$.

Thank you for listening