On Intrinsically Knotted or Completely 3-Linked Graphs

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Definition

G : finite graph

$f : G \to \mathbf{S}^3$: spatial embedding of G

We call $f(G)$ a spatial graph.

Definition

G is intrinsically linked (IL)

$\Leftrightarrow \ \forall f(G) \supset$ nonsplittable link

G is intrinsically knotted (IK)

$\Leftrightarrow \ \forall f(G) \supset$ nontrivial knot

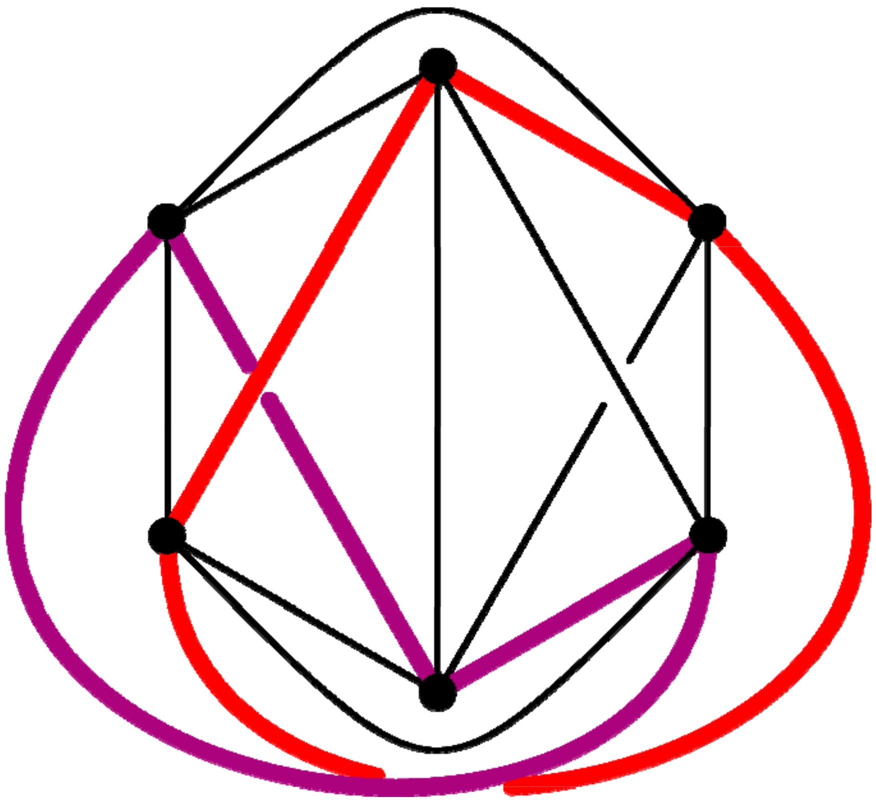
G is intrinsically 3-linked (I3L)

$\Leftrightarrow \ \forall f(G) \supset$ nonsplittable 3-component link

Intrinsically linked graph

Theorem [Sachs, Conway-Gordon]

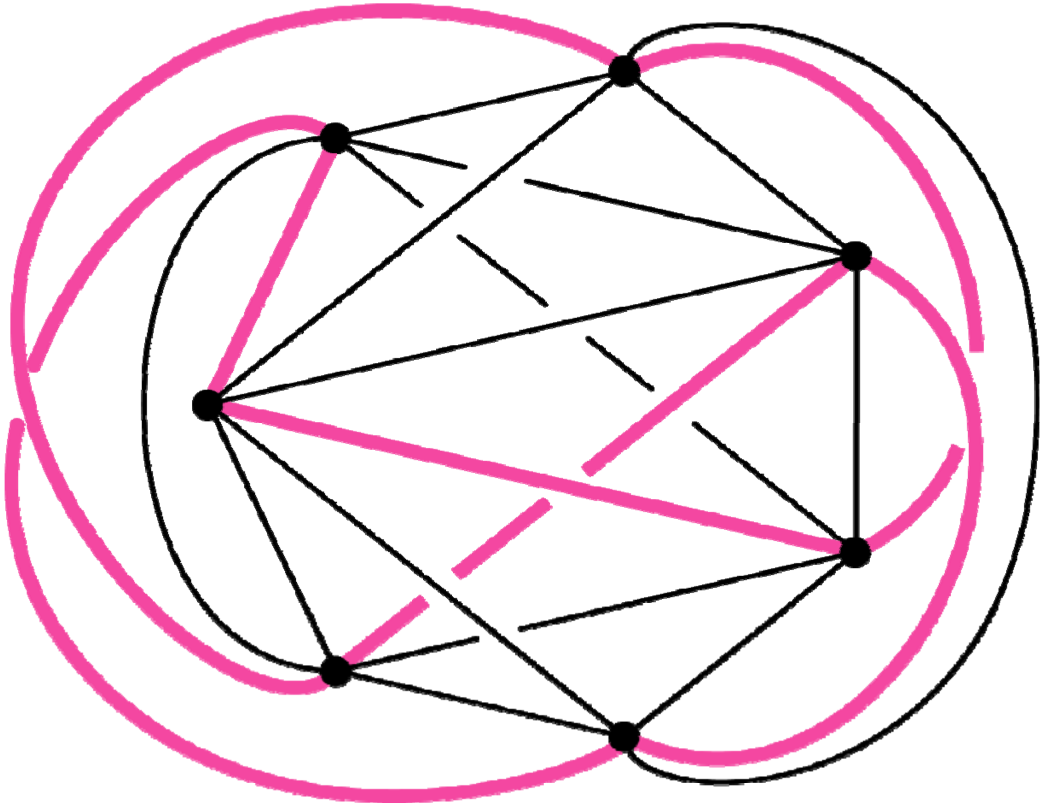
Complete graph K_6 is intrinsically linked



Intrinsically knotted graph

Theorem [Conway-Gordon]

Complete graph K_7 is intrinsically knotted



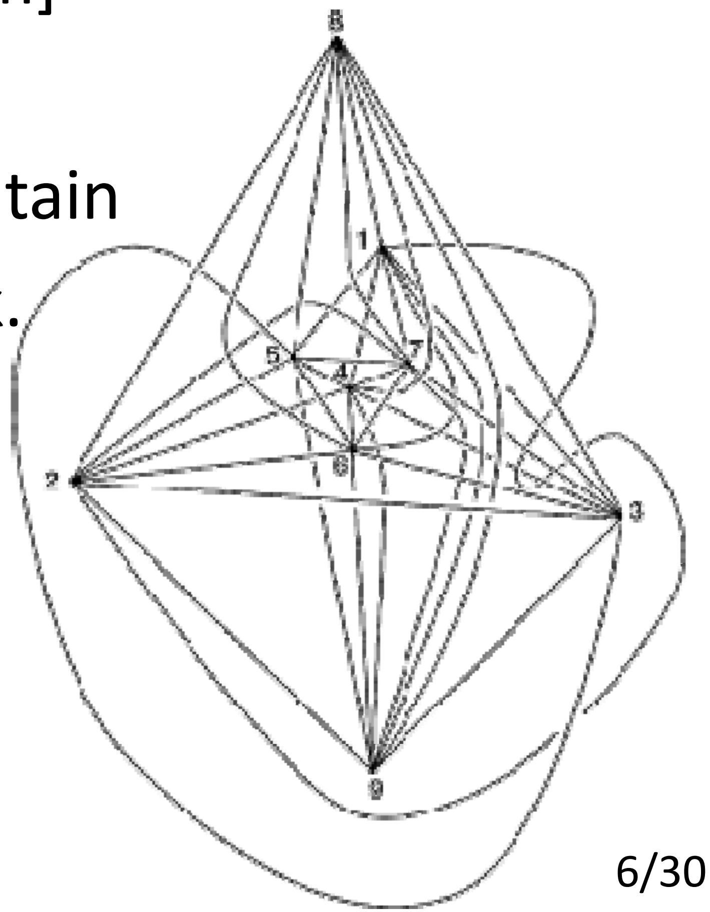
Intrinsically 3-linked graph

Theorem [Flapan-Naimi-Pommersheim]

Complete graph K_{10} is intrinsically 3-linked.

Note [Flapan-Naimi-Pommersheim]

Complete graph K_9 has a spatial embedding which does not contain nonsplittable 3-component link. Namely, K_9 is not intrinsically 3-linked.

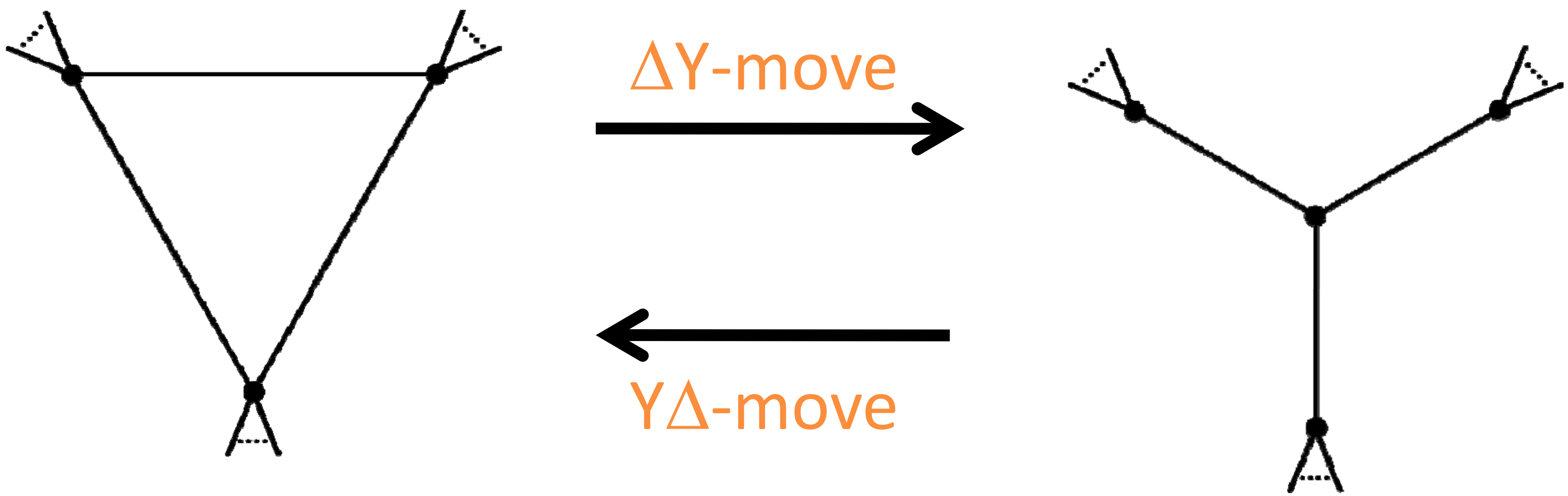


ΔY- and YΔ-moves

Definition

A ΔY-move on a graph consists of removing the edges of a 3-cycle, and adding a new vertex adjacent to the three vertices of the 3-cycle.

A YΔ-move is the inverse transformation.



ΔY-move and IK, IL and I3L

Remark

$G \rightarrow G'$; ΔY-move

G is intrinsically linked

$\Rightarrow$ G' is also intrinsically linked.

G is intrinsically knotted

$\Rightarrow$ G' is also intrinsically knotted.

G is intrinsically 3-linked

$\Rightarrow$ G' is also intrinsically 3-linked.

Namely, ΔY-move preserves intrinsically linkedness, intrinsically knottedness, and intrinsically 3-linkedness.

Characterization on Intrinsically linkedness

Theorem (Sachs' linkless embedding conjecture)
[Robertson-Seymour-Thomas]

G is intrinsically linked

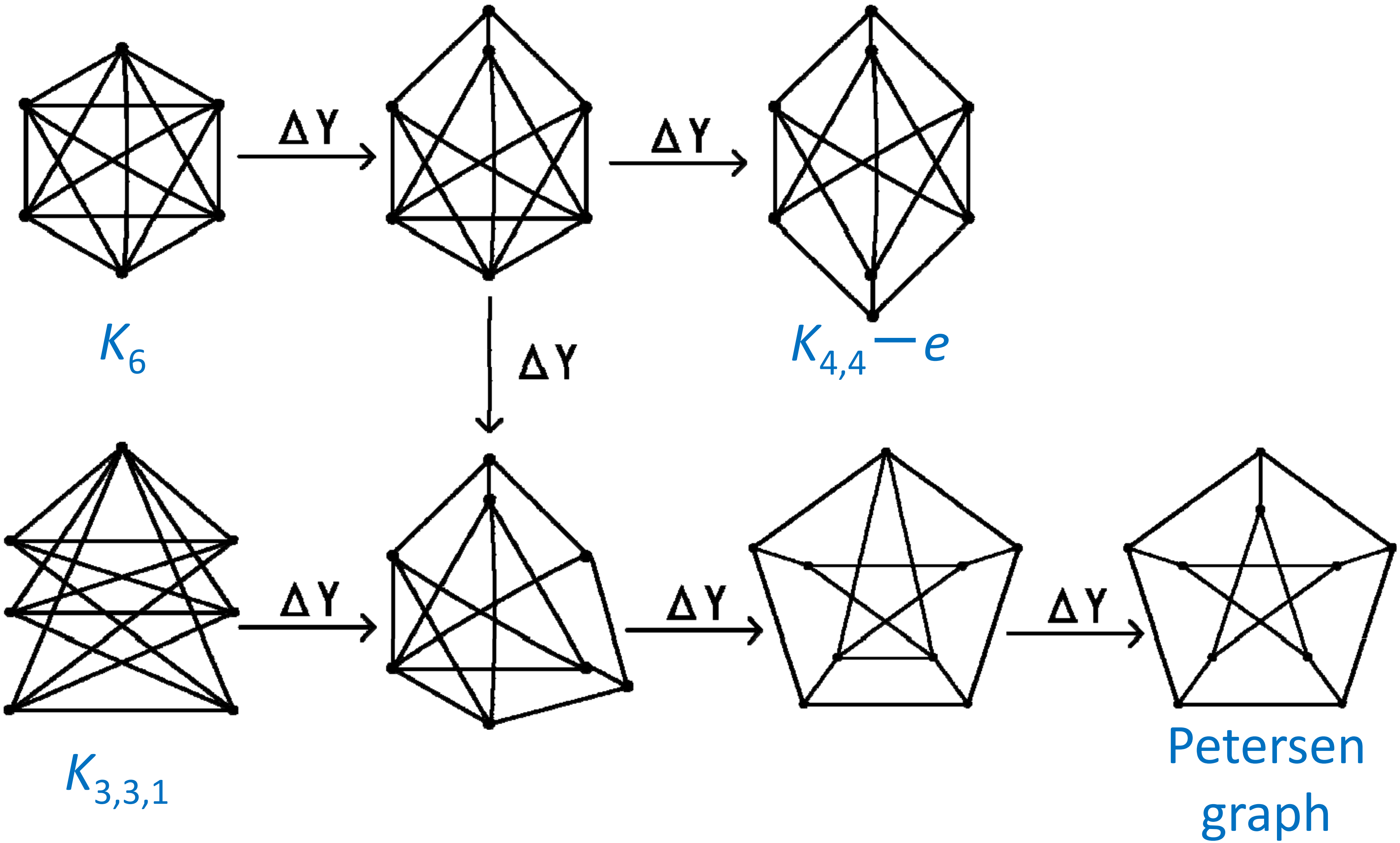
⇔ G contains a graph obtained from K_6 by ΔY- and YΔ-moves as a minor

(i.e. G contains a graph of Petersen family as a minor)

Note

The set of graphs obtained from K_6 by ΔY- and YΔ-moves is called Petersen family .

Petersen family



YΔ-move on Intrinsically linked graphs

Corollary [Robertson-Seymour-Thomas]

$G \to G'$; YΔ-move

G is intrinsically linked

$\Rightarrow G'$ is also intrinsically linked

Note

ΔY- and YΔ-move preserve intrinsically linkedness.

Flapan-Naimi's Theorem

Theorem [Flapan-Naimi]

YΔ-move does not preserve intrinsically knottedness

Note

ΔY-move preserves intrinsically knottedness

Flapan-Naimi showed that

∃ *FN* : the graph obtained from K_7 by ΔY-moves and two YΔ-moves s.t. *FN* is not intrinsically knotted

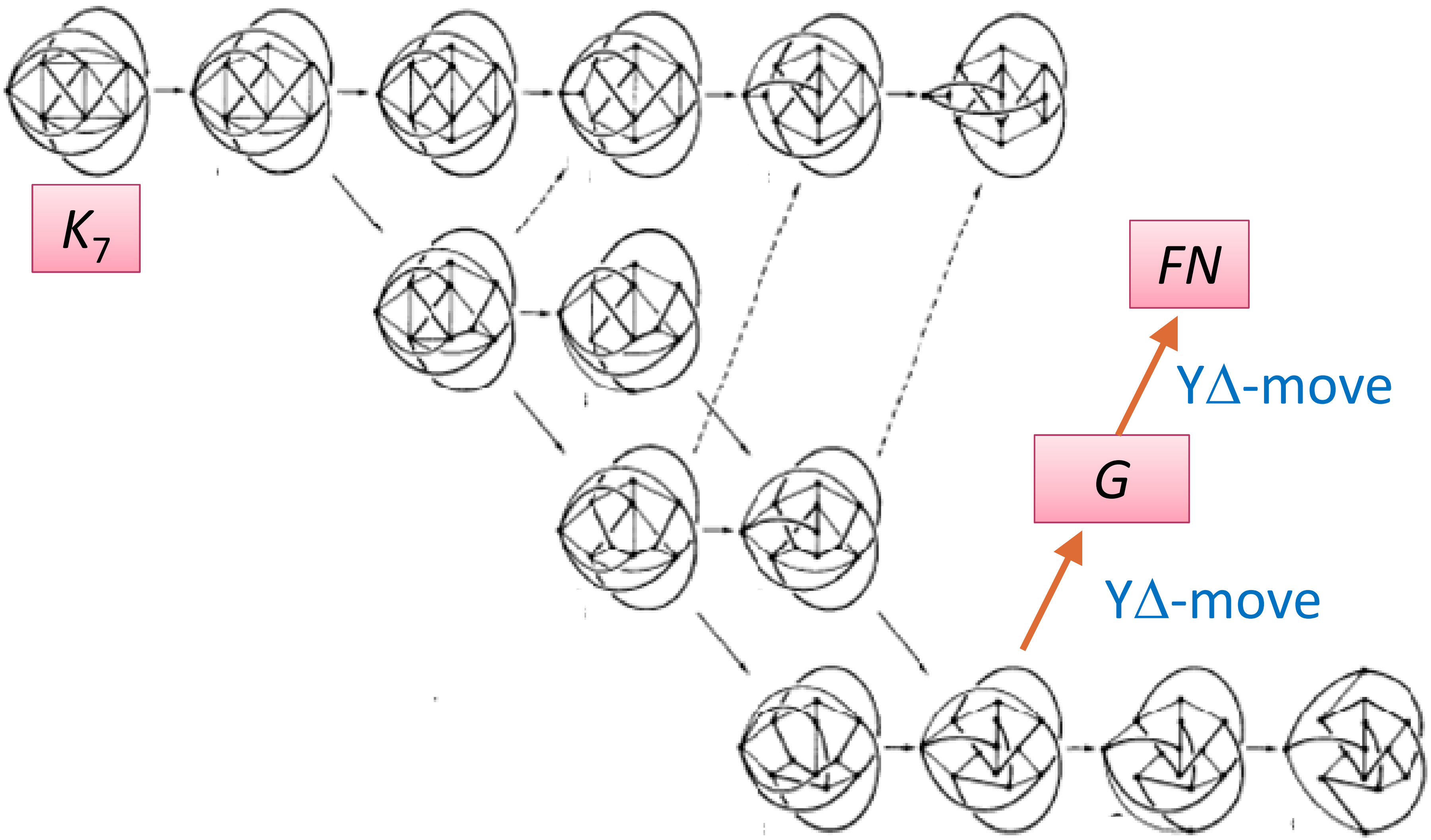
Graphs obtained from K_7 by ΔY-moves

Theorem [Kohara-Suzuki]

The thirteen graphs are obtained from K_7 by ΔY-moves.

In addition, these graphs are minor minimal w. r. t. intrinsically knottedness.

Graphs obtained from K_7 by ΔY-moves



Motivation

- Is a graph obtained from K_7 by ΔY-moves and one YΔ-move intrinsically knotted?
- How many graphs are obtained from K_7 by ΔY-and YΔ-moves?
  In addition, is each of these graphs intrinsically knotted?

Main Theorem

Theorem [H-Nikkuni-Taniyama-Yamazaki]

The nineteen graphs obtained from K_7 by ΔY- and YΔ-moves are intrinsically knotted or completely 3-linked. In addition, these graphs are minor minimal w. r. t. intrinsically knotted or completely 3-linkedness.

Definition

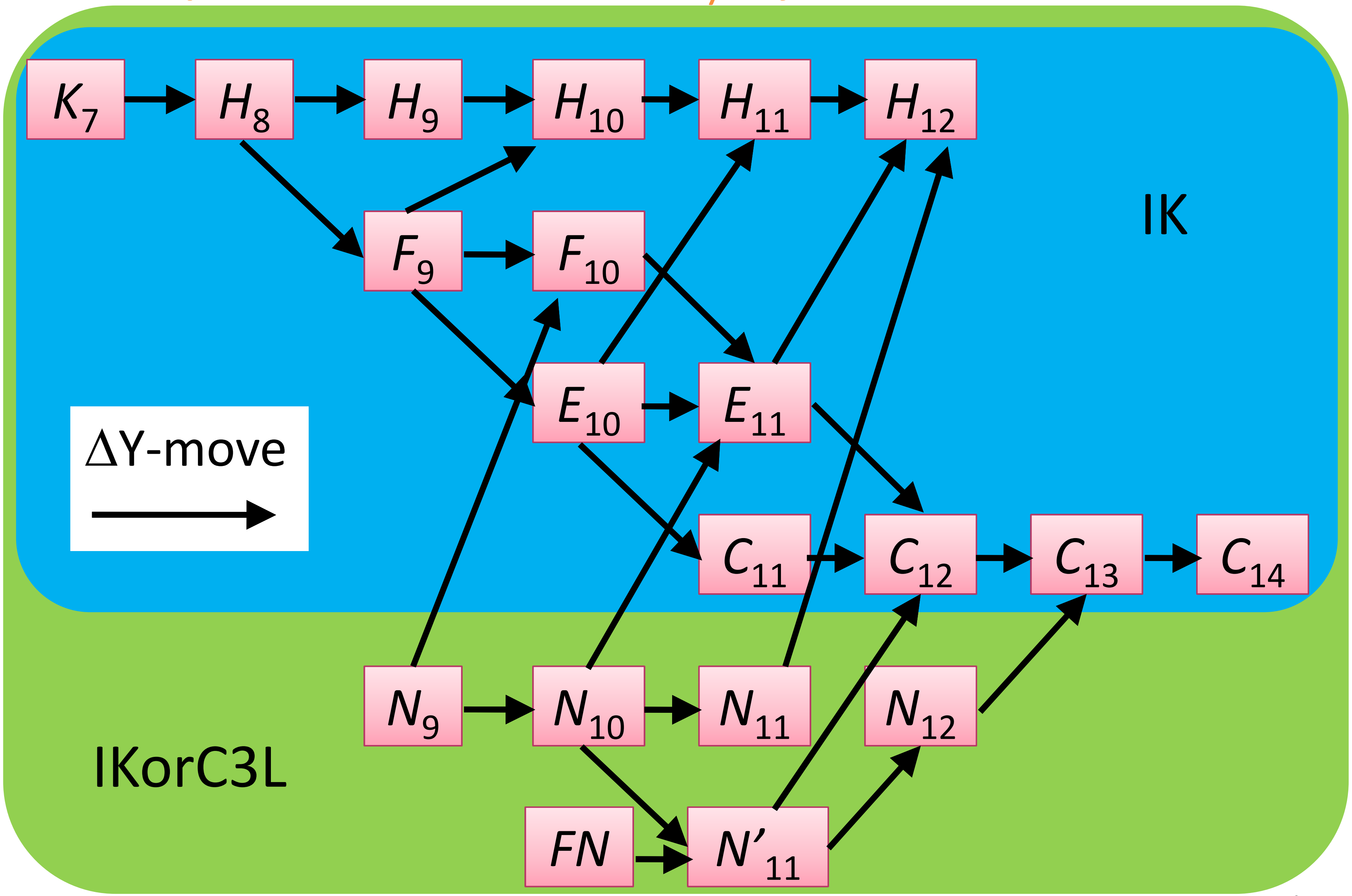
G is intrinsically knotted or completely 3-linked (IKorC3L)

$\Leftrightarrow \forall f(G) \supset$ nontrivial knot or 3-component link each of whose 2-component sublink is nonsplittable

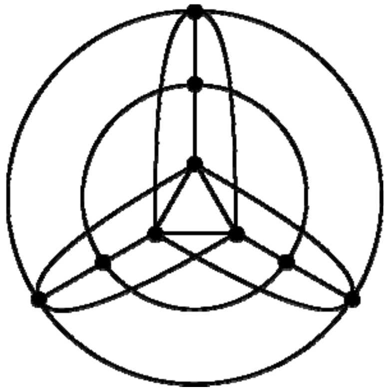
Note G is intrinsically knotted

$\Rightarrow$ G is intrinsically knotted or completely 3-linked

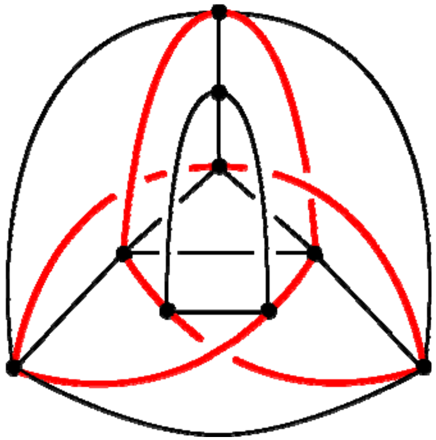
Graphs obtained from K_7 by ΔY- and YΔ-moves

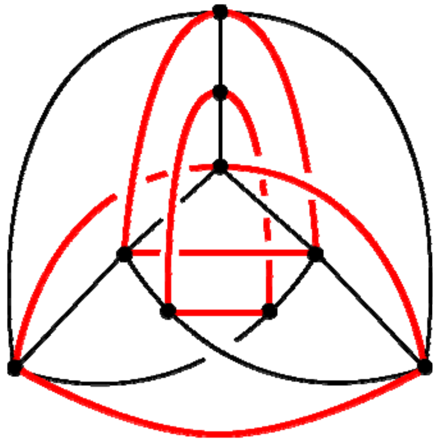


Intrinsically knotted or completely 3-linked graph



N_9





Corollary

Note

Each of N_9, N_{10}, N_{11}, N'_{11}, N_{12} and FN is neither intrinsically knotted nor intrinsically completely 3-linked.

Corollary [H-Nikkuni-Taniyama-Yamazaki]

G : graph obtained from K_7 by ΔY- and YΔ-moves but not obtained from K_7 by ΔY-moves

$\Rightarrow$ G is not intrinsically knotted

ΔY-move on IKorC3L

<u>Proposition</u>

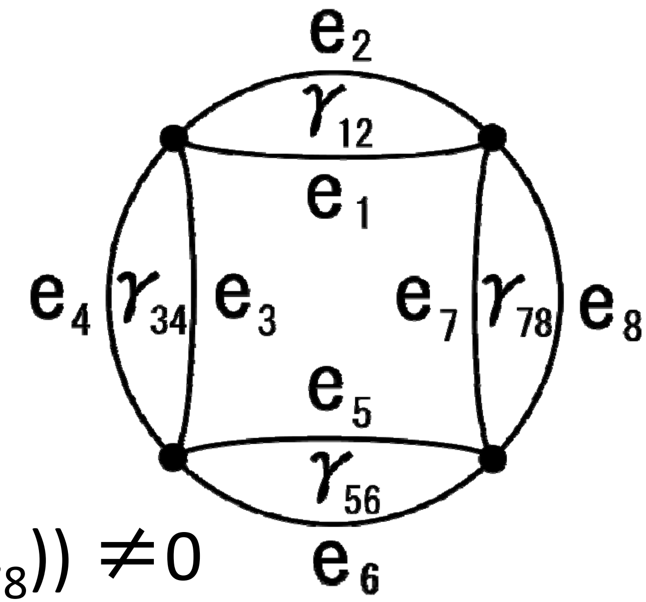
$G \to G'$; ΔY-move

G is intrinsically knotted or completely 3-linked

$\Rightarrow$ G' is also intrinsically knotted or completely 3-linked

Then we show that N_9 and FN are intrinsically knotted or completely 3-linked.

Tool of Proof



Lemma [Taniyama-Yasuhara, Foisy]

f : a spatial embedding of D_4

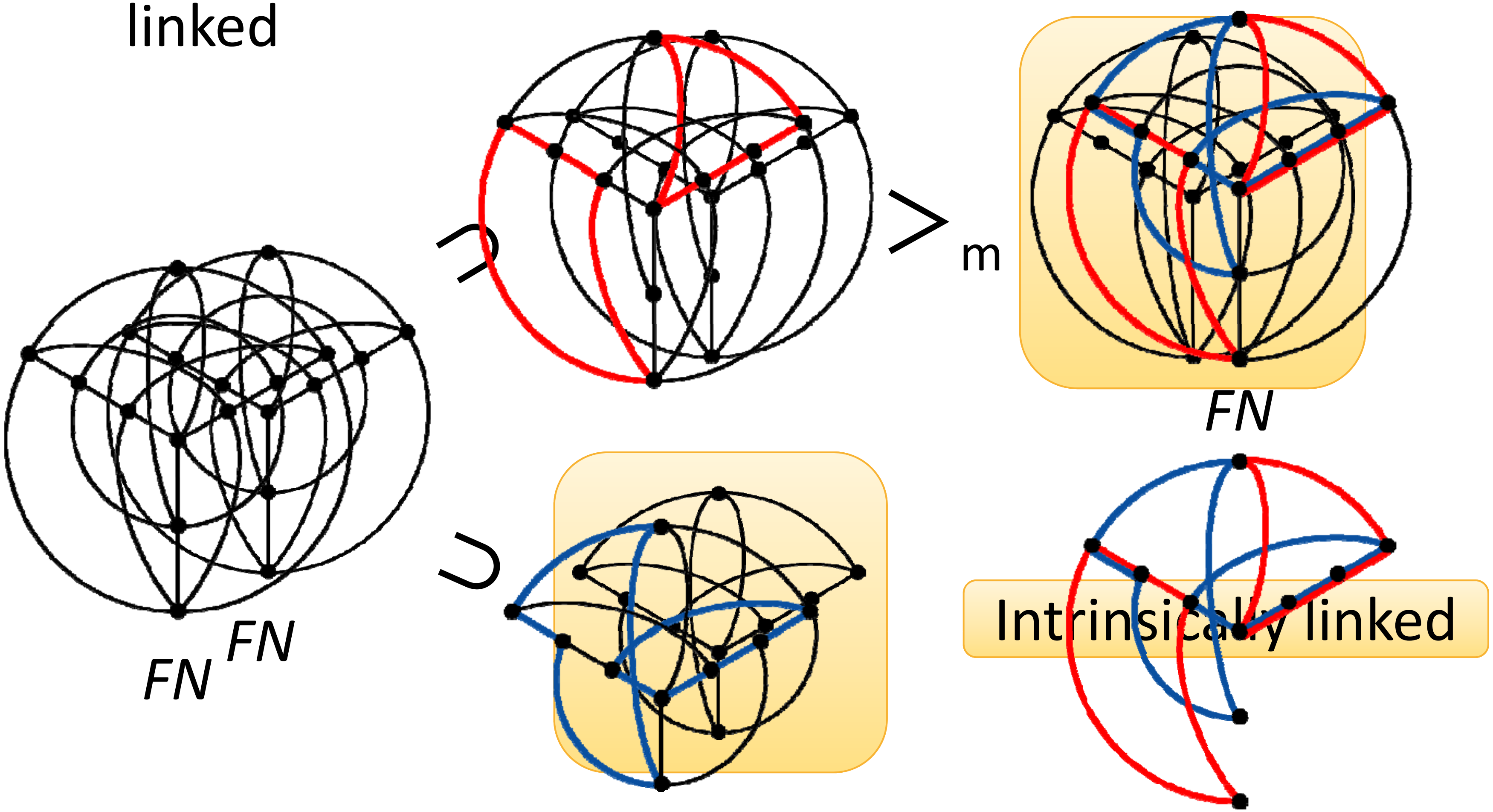
$$\sum_{4\text{-cycle}\,\gamma\subset D_4} a_2(f(\gamma)) \equiv 1 \pmod 2$$

$\Leftrightarrow \quad \mathrm{lk}_2(f(\gamma_{12} \cup \gamma_{56})) \times \mathrm{lk}_2(f(\gamma_{34} \cup \gamma_{78})) \neq 0$

where a_2 is the second coefficient of the Conway polynomial and lk_2 is mod 2 linking number.

Sketch of Proof

- Proof that *FN* is intrinsically knotted or completely 3-linked



Proposition on Minor minimality

Proposition [H-Nikkuni-Taniyama-Yamazaki]

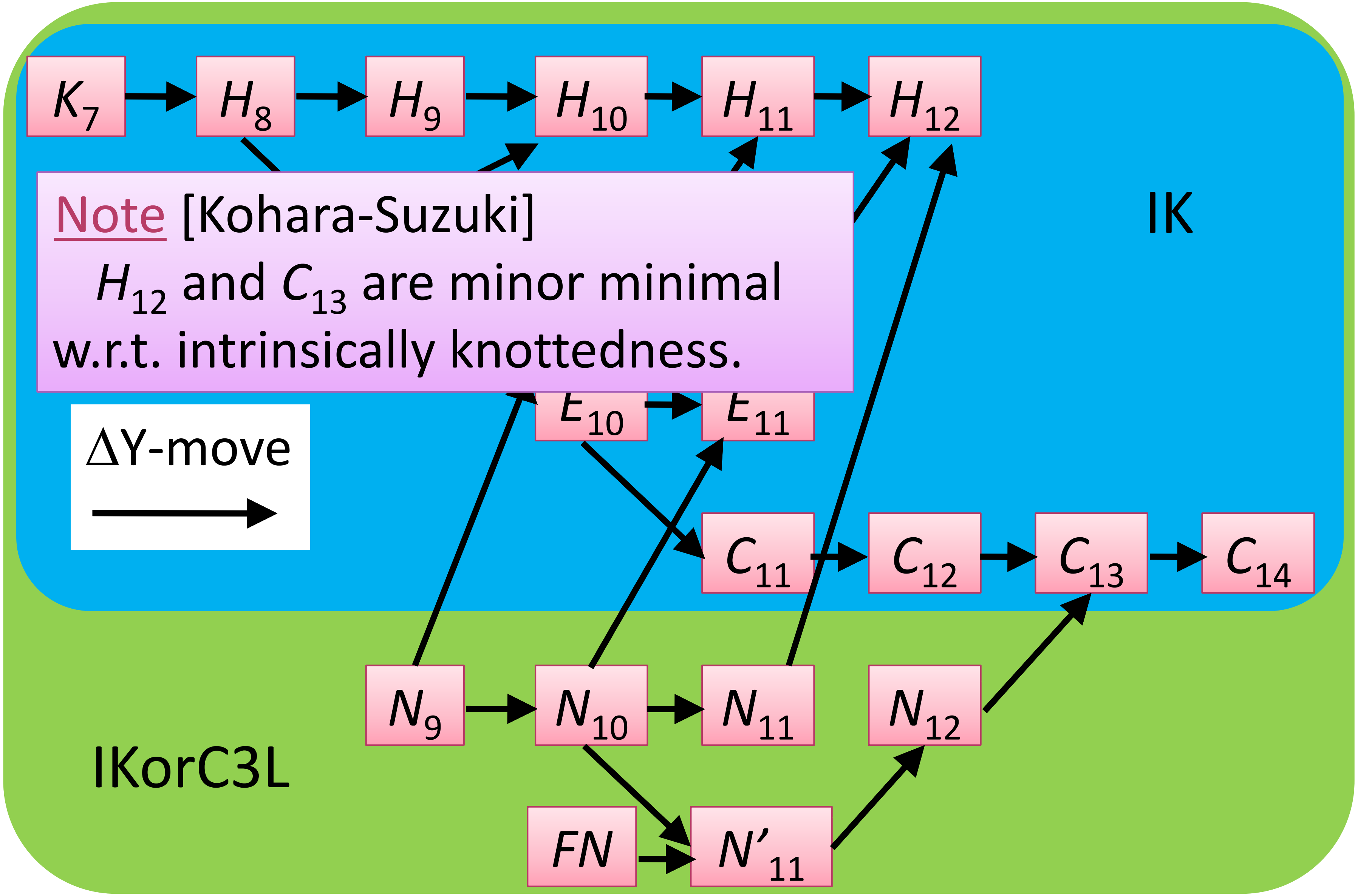
G, *G*’ : intrinsically knotted or completely 3-linked graphs

G’ is obtained from *G* by a ΔY-move

G’ is minor minimal w.r.t. intrinsically knotted or completely 3-linkedness

⇒ *G* is minor minimal w.r.t. intrinsically knotted or completely 3-linkedness

On Minor minimal w.r.t. IKorC3L



Note [Kohara-Suzuki]

H_{12} and C_{13} are minor minimal w.r.t. intrinsically knottedness.

Intrinsically knotted or 3-linked

<u>Definition</u> [Foisy]

G is intrinsically knotted or 3-linked (IKor3L)

$\Leftrightarrow \forall f(G) \supset$ nontrivial knot <u>or</u> nonsplittable 3-component link

<u>Proposition</u>

ΔY-move preserves IKor3L

<u>Note</u>

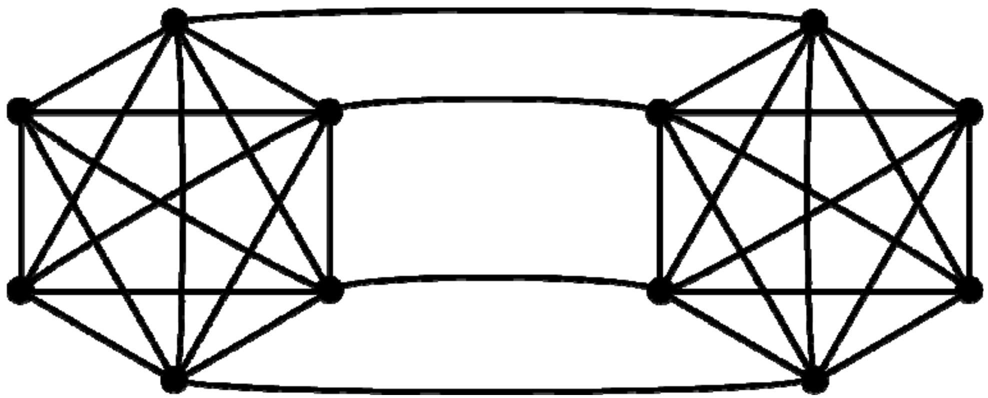
G is intrinsically knotted or completely 3-linked

$\Rightarrow$ G is intrinsically knotted or 3-linked

Results on Intrinsically knotted or 3-linked

Theorem [Foisy]

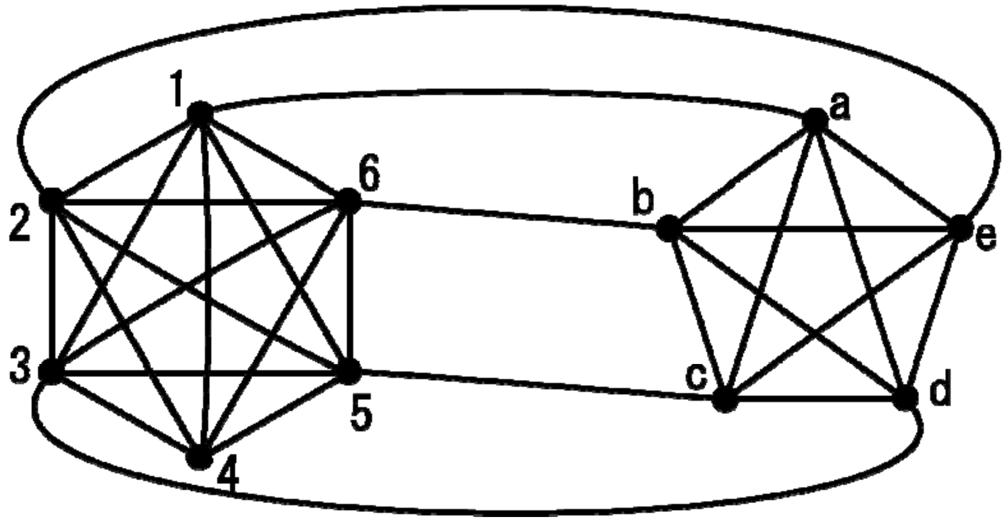
$K_6 *_4 K_6$ is intrinsically knotted or 3-linked but is neither intrinsically knotted nor intrinsically 3-linked.



Results on Intrinsically knotted or 3-linked

Theorem [Yamazaki]

$K_6 *_5 K_5$ is IKor3L but is neither IK nor I3L, and minor minimal w.r.t. IKor3L.



Proposition [Yamazaki]

YΔ-move does not preserve intrinsically knotted or 3-linked.

Note on Intrinsically knotted or 3-linked

Note

Each of $K_6 *_4 K_6$ and $K_6 *_5 K_5$ is not IKorC3L.

Example

The following spatial embedding of $K_6 *_5 K_5$ does not contain nontrivial knot and 3-component link each of whose 2-component sublink is nonsplittable.

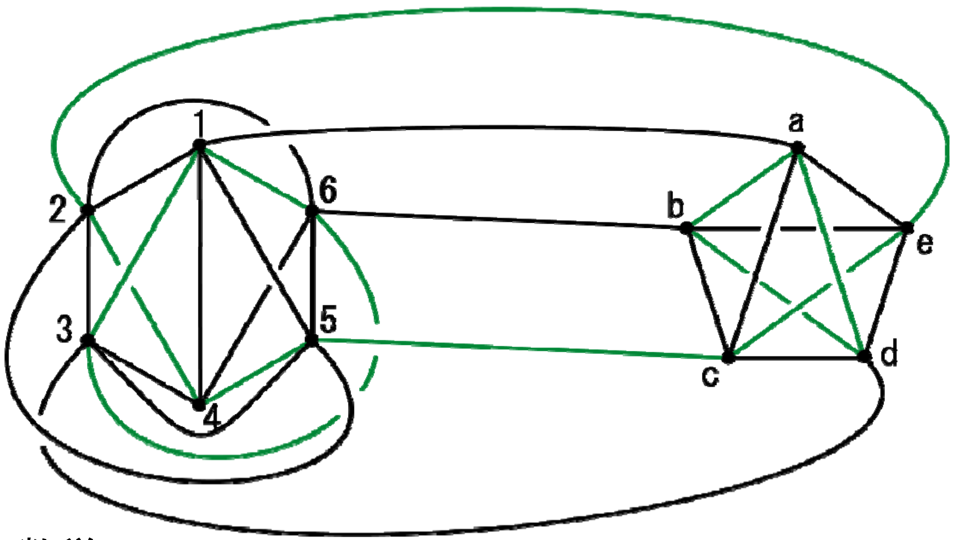


Table on moves

intrinsically	ΔY-move	YΔ-move
Linked	○	○ [Robertson-Seymour-Thomas]
Knotted	○	× [Flapan-Naimi]
3-linked	○	?
Knotted or 3-linked	○	× [Yamazaki]
Knotted or completely 3-linked	○	?

○…preserve　×…not preserve　?…unknown

Questions

- Does a YΔ-move preserve intrinsically knotted or completely 3-linkedness?
- It is known that $K_{3,3,1,1}$ is intrinsically knotted.
  Is each of the graphs obtained from $K_{3,3,1,1}$ by ΔY- and YΔ-moves intrinsically knotted or completely 3-linked?

Thank you for listening