Pseudo diagrams of knots and its related topics

Ryo Hanaki
Graduate School of Education, Waseda University

Dedicated to Professor Kunio Murasugi for his 80th birthday

1 Introduction

First, we give the definitions of pseudo diagram and the trivializing number and results on it. Next, we define the trivializing number for knots and introduce results.

1.1 Definition

We consider knots in $\mathbf{R^3}$. Let p be a natural projection from $\mathbf{R^3}$ to $\mathbf{R^2}$. We say that p is a *projection* of K if the multiple points of $p|_K$ are only finitely many transversal double points. Then we call $p(K)$ a *(knot) projection* and denote it by $P = p(K)$. See Fig. 1. A *diagram* D is a projection P with over/under information at every double point. A diagram D uniquely represents a knot up to ambient isotopy.

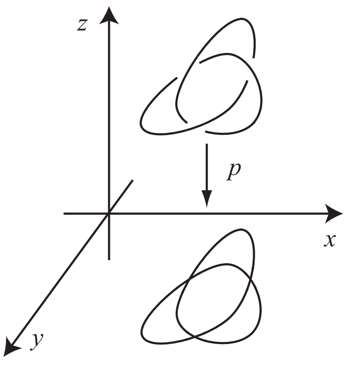

Figure 1: Knot projection

We have the following question.

Question 1 *Can we determine from P whether the original knot is trivial or knotted?*

We cannot determine it except some special cases. Because we do not know over/under information at each double point of P. For example, let P be a projection of a knot with 3 double points as illustrated in Fig. 2. Then we have 2^3 diagrams obtained from P. Two diagrams represent nontrivial knots and six diagrams represent the trivial knots.

Figure 2: Projection and diagrams obtained from it

Then we have a natural question.

Question 2 *Which double points of a projection and which over/under informations at them should we know in order to determine that the original knot is trivial or knotted?*

We introduced a notion of the pseudo diagram in [2]. In this paper, a double point with over/under information is called a *crossing*, in contrast a double point without over/under information is called a *pre-crossing*. We say that a *pseudo diagram* Q is a projection P with over/under information at some pre-crossings of P. Here, we allow the possibility that a pseudo diagram is a projection or a diagram. Let Q and Q' be pseudo diagrams of a projection. Then we say that *a pseudo diagram* Q' *is obtained from a pseudo diagram* Q if each crossing of Q has the same over/under information with Q'. A pseudo diagram Q is said to be *trivial* if for any diagram D obtained from Q, D represents the trivial knot. For example, in Fig. 3, (a) is trivial, both (b) and (c) are not trivial.

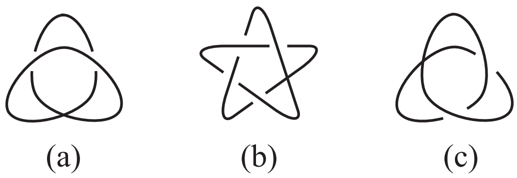

Figure 3: Pseudo diagrams

We define that the *trivializing number of* P is the minimal number of the crossings of Q where Q varies over all trivial pseudo diagrams obtained from P. We denote the trivializing number of P by $\mathrm{tr}(P)$. In [2], similarly we gave the definitions of the knotted pseudo diagram and the knotting number and study it. In this paper, we narrow focus on trivializing number. For example, let P be the projection as illustrated in Fig. 2, then $\mathrm{tr}(P) = 2$.

1.2 Results on pseudo diagrams

We introduce the results on trivial pseudo diagrams and a method for calculating the trivializing number of a projection.

Theorem 1 *For any projection P, the trivializing number of P is even.*

Proposition 2 *For any non-negative even number n, there exists a projection P with* $\mathrm{tr}(P) = n$.

For example, the projection as illustrated in Fig. 4 where $m = n + 1$ has trivializing number n.

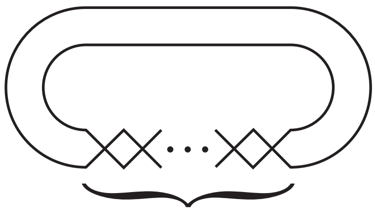

Figure 4: Pseudo diagrams

We introduce a chord diagram to calculate the trivializing number of a projection. Let Q be a pseudo diagram with n pre-crossings. A *chord diagram* of Q is a circle with n chords marked on it by dashed line segment where the preimage of each pre-crossing is connected by a chord. We denote it by CD_Q. For example, let Q be a pseudo diagram (a) in Fig. 5. Then a chord diagram (b) in Fig. 5 is CD_Q. Note that for each chord of a chord diagram of a projection, each of the two arcs in the circle bounded by the end points of the chord contains even number of end points of the other chords.

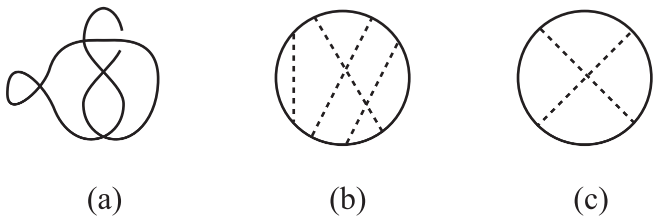

Figure 5: Chord diagram.

We have the following lemmas.

Lemma 3 *Let Q be a pseudo diagram such that CD_Q contains a sub-chord diagram as Fig. 5(c). Then Q is not trivial.*

Proof. Let Q' be a pseudo diagram obtained from Q such that $CD_{Q'}$ is the chord diagram as Fig. 5(c). Let p_1, p_2 be the pre-crossings of Q'. We give an orientation to a circle. Let K_1, K_2, K_3 and K_4 be the knots represented by D_{++}, D_{+-}, D_{-+} and D_{--} respectively where D_{+-} is obtained from Q' by changing p_1 to be a positive crossing and p_2 to be a negative crossing. Note that these knots are obtained from Q. Then, $a_2(K_1) - a_2(K_2) - a_2(K_3) + a_2(K_4) = 1$ holds where a_2 is the second coefficient of the Conway polynomial. At least one of K_1, K_2, K_3 and K_4 is nontrivial since a_2 of the trivial knot is zero. Therefore, Q is not trivial. □

Lemma 4 *Let P be a projection. Let CD be a sub-chord diagram of CD_P such that CD does not contain a sub-chord diagram as Fig. 5(c). Then there exists a trivial pseudo diagram Q obtained from P such that $CD_Q = CD$.*

Proof. Let p_1 be a pre-crossing of P which corresponds to an outer most chord c_1 in CD and l_1 the sub-arc on P which corresponds to the outer most arc. By giving over/under information to each pre-crossing on l_1 so that l_1 goes over the others as in Fig. 6, we obtain a pseudo diagram Q_1 from P. Next, let p_2 be a pre-crossing of Q_1 which corresponds to an outer most chord c_2 under forgetting c_1 in CD, and l_2 the sub-arc on Q_1 which corresponds to the outer most arc. By giving over/under information to each pre-crossing on l_2 so that l_2 goes over the others except l_1, we obtain a pseudo diagram Q_2 from Q_1. By repeating this procedure until all of the chords are forgotten, we obtain a pseudo diagram Q from P. For any diagram D obtained from Q, first we can vanish the crossings on l_1 and the crossing corresponding to p_1, next we can vanish the crossings on l_2 and the crossing corresponding to p_2, similarly we can vanish all crossings of D. Therefore, we see that Q is trivial. □

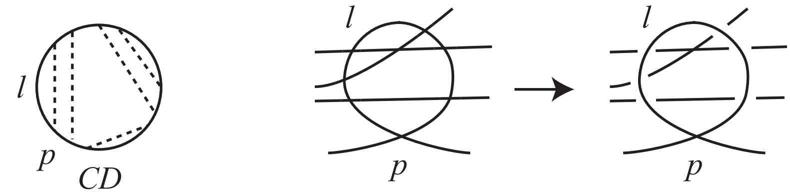

Figure 6:

For a projection P, by applying Lemmas 3 and 4, we can calculate $\mathrm{tr}(P)$ from CD_P. For example, let P be a projection as (a), then CD_P is the chord diagram as (b) in Fig. 7. Any chord diagram obtained from CD_P by deleting at most three chords contains a sub-chord diagram as Fig. 5(c). A chord diagram CD obtained from CD_P by deleting four chords as (c) in Fig. 7 does not contain a sub-chord diagram as Fig. 5(c). Therefore, we get $\mathrm{tr}(P) = 4$ and a pseudo diagram (d) in Fig. 7 is a trivial pseudo diagram which realizes the trivializing number of P.

We see from Lemmas 3 and 4 and the property of a knot projection that Theorem 1 holds.

We have the following theorems by applying Lemmas 3 and 4.

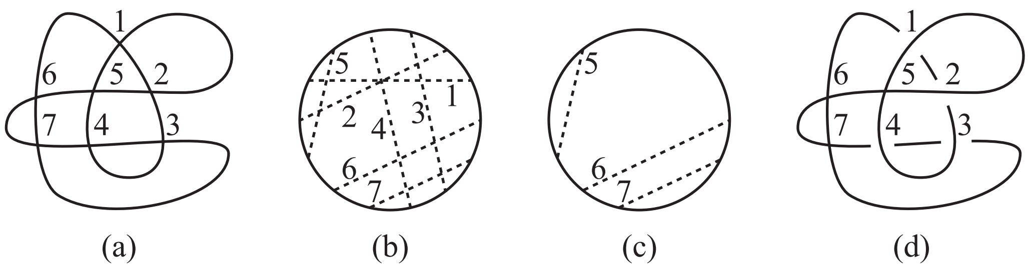

Figure 7:

Theorem 5 *Let P be a projection. Then* $\mathrm{tr}(P) = 2$ *if and only if P is obtained from the projection as illustrated in Fig. 8 (a) where m is a positive integer by a series of replacing a sub-arc of P as illustrated in Fig. 8 (b).*

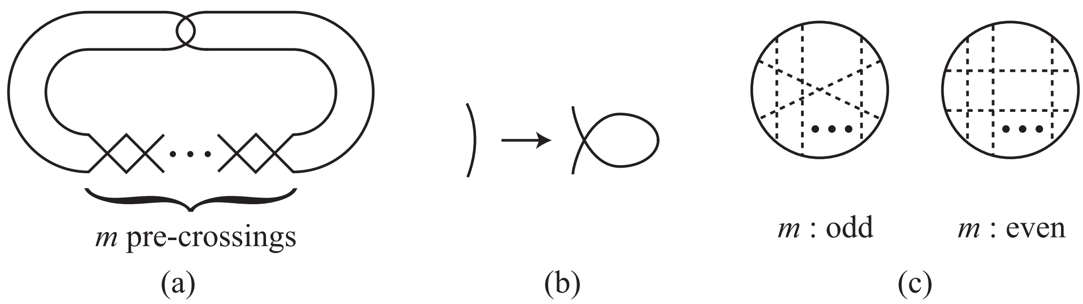

Figure 8:

Theorem 6 *Let P be a projection with at least one pre-crossing. Then it holds that* $\mathrm{tr}(P) \leq p(P) - 1$. *The equality holds if and only if P is one of the projections as illustrated in Fig. 4 where m is a positive odd integer.*

1.3 On pseudo diagrams for virtual knots

Recently, A. Henrich etc. expanded pseudo diagrams for virtual knots in [4]. They discuss relation between trivializing number and unknotting number (resp. genus) in the paper.

2 Trivializing number for knots

We define the trivializing number for knots and study it.

Let K be a knot, we define the following:

$$\mathrm{tr}(K) = \min\{\mathrm{tr}(P) | \text{A diagram obtained from a projection } P \text{ represents } K\}.$$

Then we call $\mathrm{tr}(K)$ the *trivializing number of K*. We see from Theorem 1 that $\mathrm{tr}(K)$ is even for any knot K. Henrich-etc. provide a table of trivializing numbers for knots with up to 10 crossings. However, we do not know the trivializing number for all knots with up to 10 crossings. The following proposition holds.

Proposition 7 [3, 4] *Let K be a knot. Then $u(K) \leq \dfrac{\mathrm{tr}(K)}{2}$ holds where $u(K)$ is the unknotting number of K.*

Proof. It follows from the definition of the trivializing number and a fact that a mirror diagram of a trivial knot is also trivial. □

Similarly, it is known in [4] that the following relation between the genus and the trivializing number holds.

Theorem 8 [4] *Let K be a knot. Then $g(K) \leq \dfrac{\mathrm{tr}(K)}{2}$ holds where $g(K)$ is the genus of K.*

Proposition 9 *For any non-negative integer n, there exists an alternating knot K such that $\dfrac{\mathrm{tr}(K)}{2} - u(K) = n$.*

Proof. Let D_0 be the diagram as illustrated in Fig. 9. In the case $n = 0$, let D'_0 be the alternating diagram obtained from D_0 by crossing change at the crossing framed by a dash circle in Fig. 9. Let K_0 be the knot represented by D'_0. Since K_0 is an alternating knot and it is known in [5, 1] that an orientable surface obtained by Seifert algorithm in an alternating diagram realizes the minimal genus, we get $g(K_0) = 1$. We see from Theorem 8 and a chord diagram of D_0 that $\mathrm{tr}(K_0) = 2$. It is obvious that $u(K_0) = 1$. Therefore, $\dfrac{\mathrm{tr}(K_0)}{2} - u(K_0) = 0$.

In the case $n \geq 1$, let D_n be the almost alternating diagram obtained from D_0 by replacing around the crossing framed by a dash circle in Fig. 9 by (b) as illustrated in Fig. 9 n times. Then, let D'_n be the alternating diagram obtained from D_n by crossing change at the crossing framed by a dash circle. Let K_n be the knot represented by D'_n. Similarly, we see that $\mathrm{tr}(K_n) = 2(n+1)$ and $u(K_n) = 1$. Therefore, $\dfrac{\mathrm{tr}(K_n)}{2} - u(K_n) = n$. □

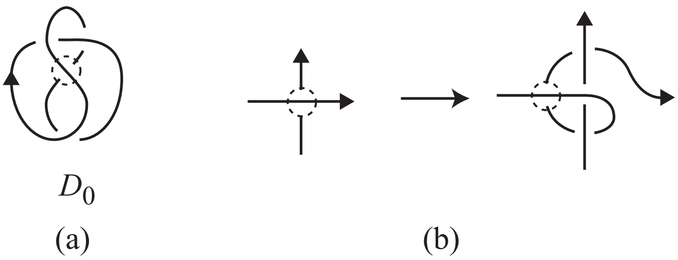

Figure 9:

We have the following theorems from the theorems for projections.

Theorem 10 *Let K be a knot. Then* $\mathrm{tr}(K) = 2$ *if and only if K is a twist knot.*

This follows from Theorem 5. Then we have the following from Theorem 6.

Theorem 11 *Let K be a nontrivial knot. Then* $\mathrm{tr}(K) \leq c(K) - 1$ *where $c(K)$ is the crossing number of K. The equality holds if and only if K is a $(2, p)$-torus knot where p is some odd number more than one.*

We have the inequality between the trivializing number and the unknotting number by Proposition 7, then we get knots which hold the equality.

Proposition 12 *Let K be a positive knot with $c(K) \leq 10$. Then* $\mathrm{tr}(K) = 2u(K)$*. Moreover, let P be the projection of a positive diagram of K, then* $\mathrm{tr}(P) = \mathrm{tr}(K)$*.*

Theorem 13 *Let K be a positive braid knot. Then* $\mathrm{tr}(K) = 2u(K)$*.*

We introduce the theorem and the proposition to estimate the unknotting number.

Theorem 14 *[6, 7] Let D be a positive diagram and K the knot represented by D. Then $2g_4(K) = 2g(K) = c(D) - O(D) + 1$ holds where $c(D)$ is the number of the crossings, $O(D)$ is the number of the Seifert circles and $g_4(K)$ is the minimum genus of a surface locally flatly embedded in the 4-ball with boundary K.*

We note that $s(K) = c(D) - O(D) + 1$ for a positive knot K and a positive diagram D of K where $s(K)$ is the Rasmussen invariant.

Proposition 15 *Let K be a knot. Then $u(K) \geq g_4(K)$.*

Proof of Theorem 13. Let D be a positive braid diagram of K. Let P be the projection of D. By Propositions 7 and 15 and Theorem 14,

$$\mathrm{tr}(P) \geq \mathrm{tr}(K) \geq 2u(K) \geq 2g_4(K) = c(D) - O(D) + 1.$$

Then, we see that $\mathrm{tr}(P) = c(D) - O(D) + 1$. Therefore, $\mathrm{tr}(K) = 2u(K)$. $\square$

We have the following question.

Question 3 *Does there exist a positive knot K with* $\mathrm{tr}(K) \neq 2u(K)$*?*

Acknowledgment

I would like to Professor Seiichi Kamada for pointing out an error of an example in a proof of Proposition 9.

References

[1] R. Crowell, *Genus of alternating link types*, Ann. of Math. (2) **69** 1959 258–275.

[2] R. Hanaki, *Pseudo diagrams of knots, links and spatial graphs*, to appear in Osaka J. of Mathematics.

[3] R. Hanaki, *Trivializing number of knots*, in preparation.

[4] A. Henrich, N. Macnaugton, S. Narayan, O. Pechenik and J. Townsend, *Classical and virtual pseudodiagram theory and new bounds on unknotting numbers and genus*, preprint, arXiv:math.GT/0908.1981.

[5] K. Murasugi, *On the genus of the alternating knot. I, II*, J. Math. Soc. Japan **10** (1958) 94–105, 235–248.

[6] T. Nakamura, *Four-genus and unknotting number of positive knots and links*, Osaka J. Math. **37** (2000), 441–451.

[7] J. Rasmussen, *Khovanov homology and the slice genus*, preprint, arXiv:math/0402131.

GRADUATE SCHOOL OF EDUCATION, WASEDA UNIVERSITY, NISHI-WASEDA 1-6-1, SHINJUKU-KU, TOKYO, 169-8050, JAPAN.

E-mail address: **r.t@fuji.waseda.jp**