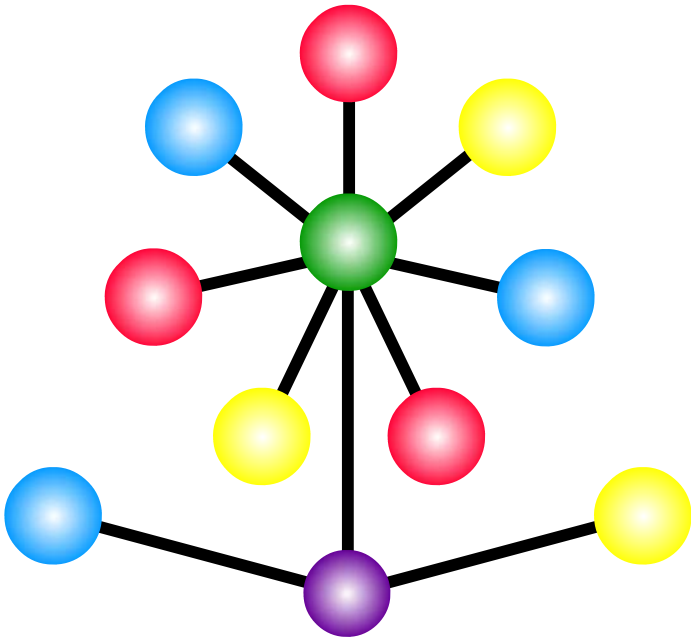
On strongly almost trivial spatial graphs



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1.1 Definitions

G : finite graph

f is a spatial embedding of G

$\overset{def.}{\Longleftrightarrow}$ $f : G \to R^3$: embedding

We call $f(G)$ a spatial graph

f, f' : spatial embeddings of G

f and f' are equivalent ($f \sim f'$)

$\overset{def.}{\Longleftrightarrow}$ $^{\exists} h : R^3 \to R^3$: orientation preserving self-homeomorphism

s.t. $h \circ f = f'$

1.1 Definitions

f is trivial

$\overset{def.}{\Longleftrightarrow}$ ${}^{\exists}f\,' \sim f$

s.t. $f\,'(G) \subset R^2 \times \{0\} \subset R^3$

G is planar

$\overset{def.}{\Longleftrightarrow}$ ${}^{\exists}f : G \longrightarrow R^2$: embedding

Hence

G has trivial spatial embeddings $\Leftrightarrow$ G is planar

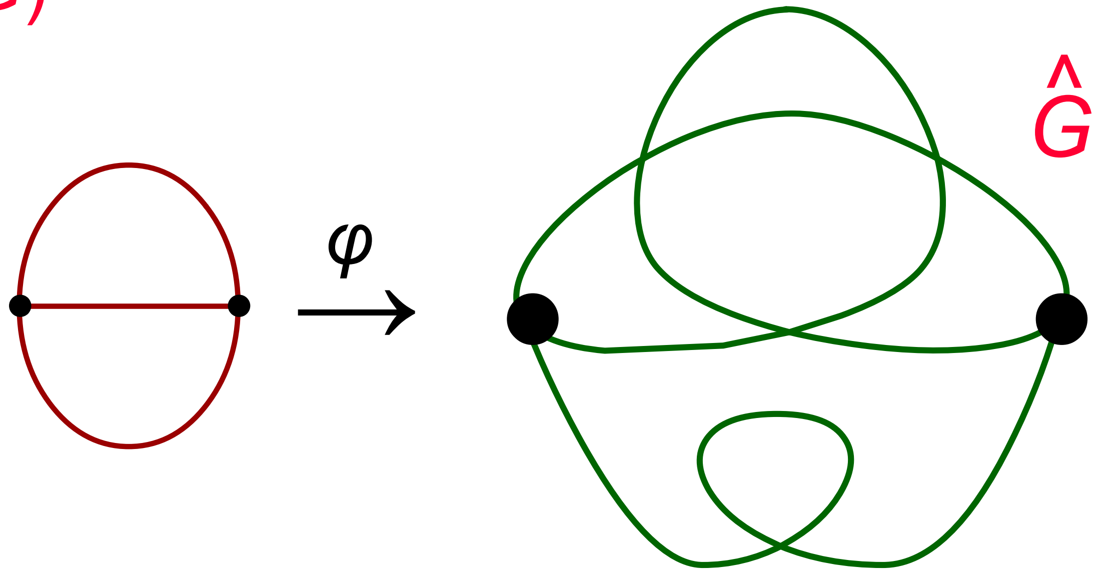
1.2 Definitions

$\varphi : G \to R^2$: continuous map

φ is a projection of G

$\overset{def.}{\Leftrightarrow}$ multiple points of φ are only finitely many transversal double points of edges

We call the image of a projection a regular projection and denote it by $\hat{G} = \varphi(G)$



1.2 Definitions

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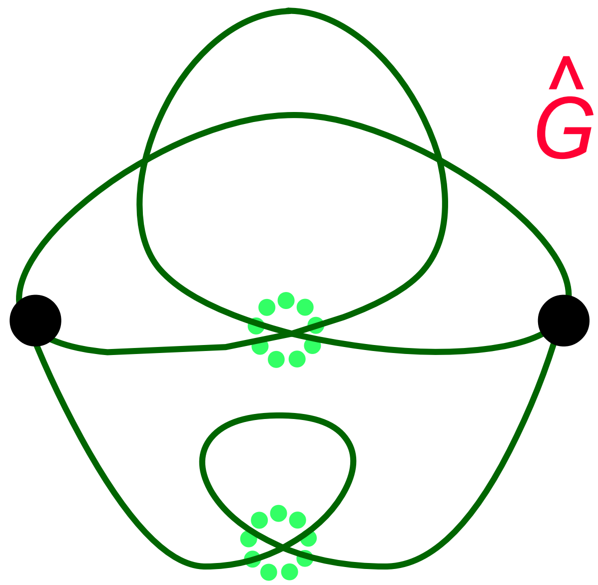


A double point of a regular projection is called a crossing

In particular,
a crossing point c is a self-crossing

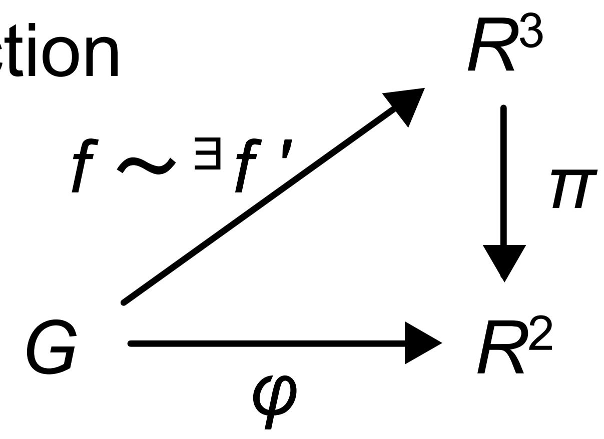
$\overset{def.}{\Leftrightarrow}$ $\varphi^{-1}(c) \subset e$, where e is an edge of G

1.2 Definitions

φ is a projection of a spatial embedding f

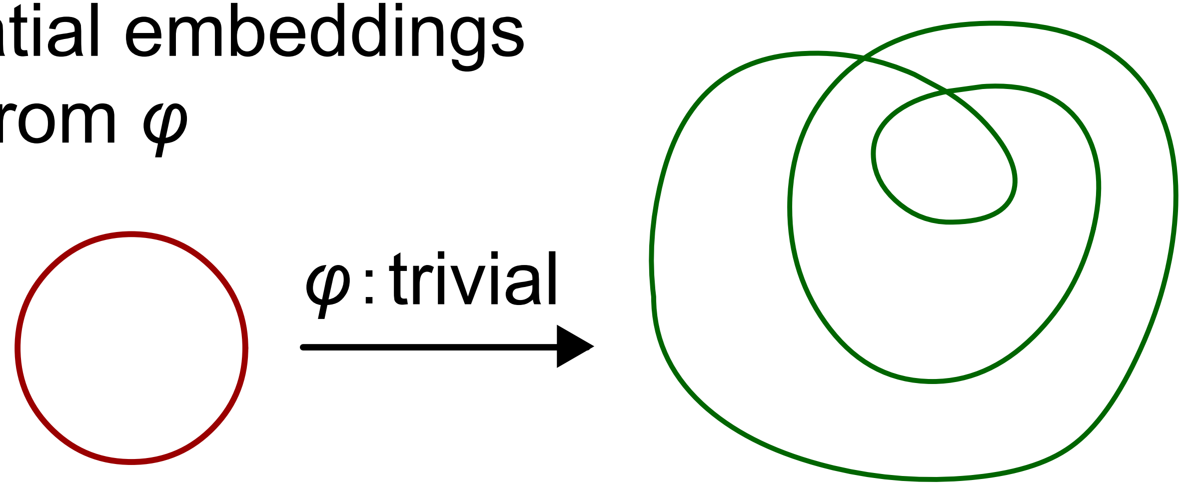
$\overset{def.}{\Leftrightarrow}$ $\exists f' \sim f$ s.t. $\varphi = \pi \circ f'$
where $\pi : R^3 \to R^2$ is a natural projection

We say f is obtained from φ



projection φ is trivial

$\overset{def.}{\Leftrightarrow}$ only trivial spatial embeddings are obtained from φ



1.3 Definitions

G : planar graph

f : spatial embedding of G

f is almost trivial.

$\overset{def.}{\Longleftrightarrow}$ ${}^{\forall}H \subsetneq G$: proper subgraph, $f|_H$ is trivial

1.3 Definitions

G : planar graph

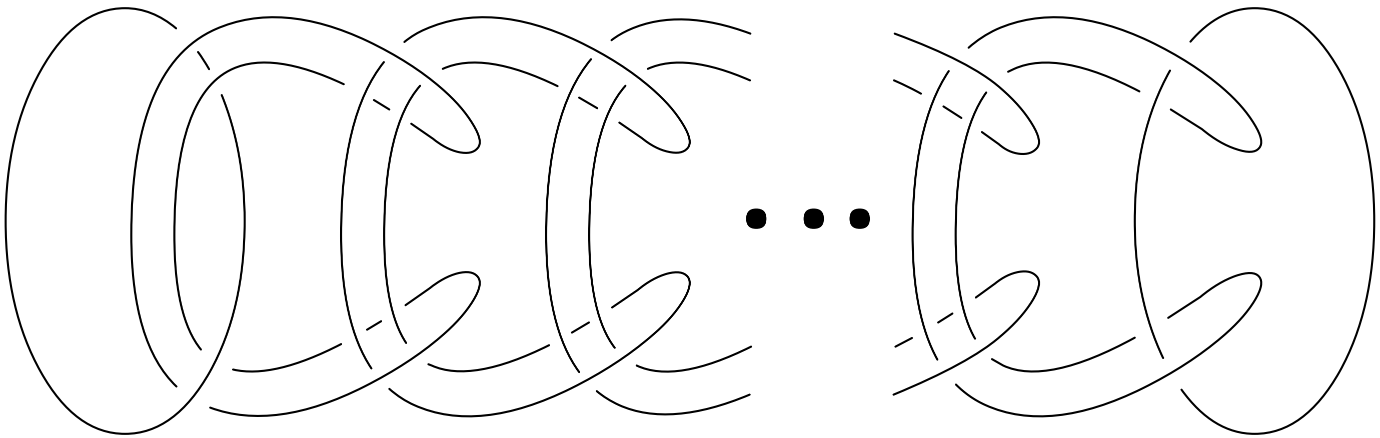
f : spatial embedding of G

f is minimally knotted.

$\overset{def.}{\Longleftrightarrow}$ f is nontrivial.

$\forall H \subsetneq G$: proper subgraph, $f|_H$ is trivial

ex.



Brunnian link

1.3 Definitions

G : planar graph, f : spatial embedding of G

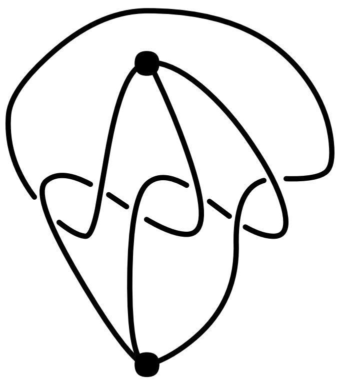
f is strongly almost trivial (SAT).

$\overset{def.}{\Leftrightarrow}$ f is nontrivial, $^{\exists}\hat{f}$: projection of f
s.t. $^{\forall}H \subsetneq G$: proper subgraph, $\hat{f}|_H$ is trivial

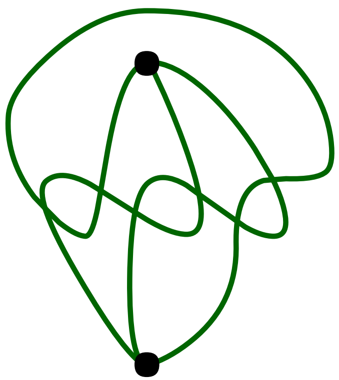
We call $\hat{f}$ SAT projection.

Hence, f is a SAT embedding of $G \Rightarrow f$ is minimally knotted

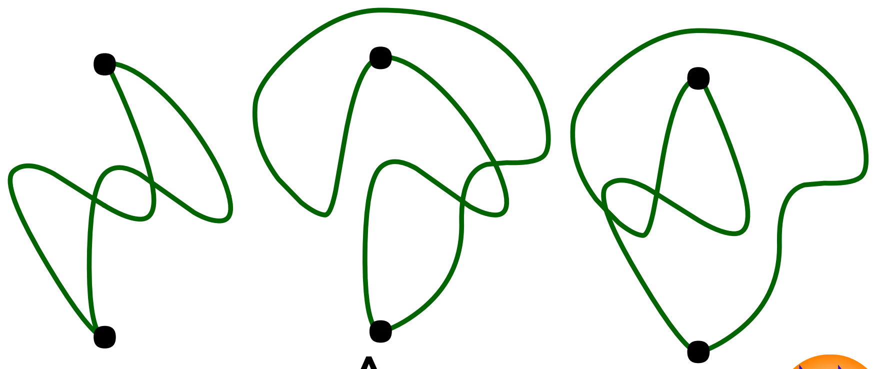
ex. θ-curve has strongly almost trivial embeddings.



f



$\hat{f}$



$\hat{f}|_H$

1 2 3 4 5 6 7 8 9 10 11

2.1 SATに関連すること

φ : projection of G

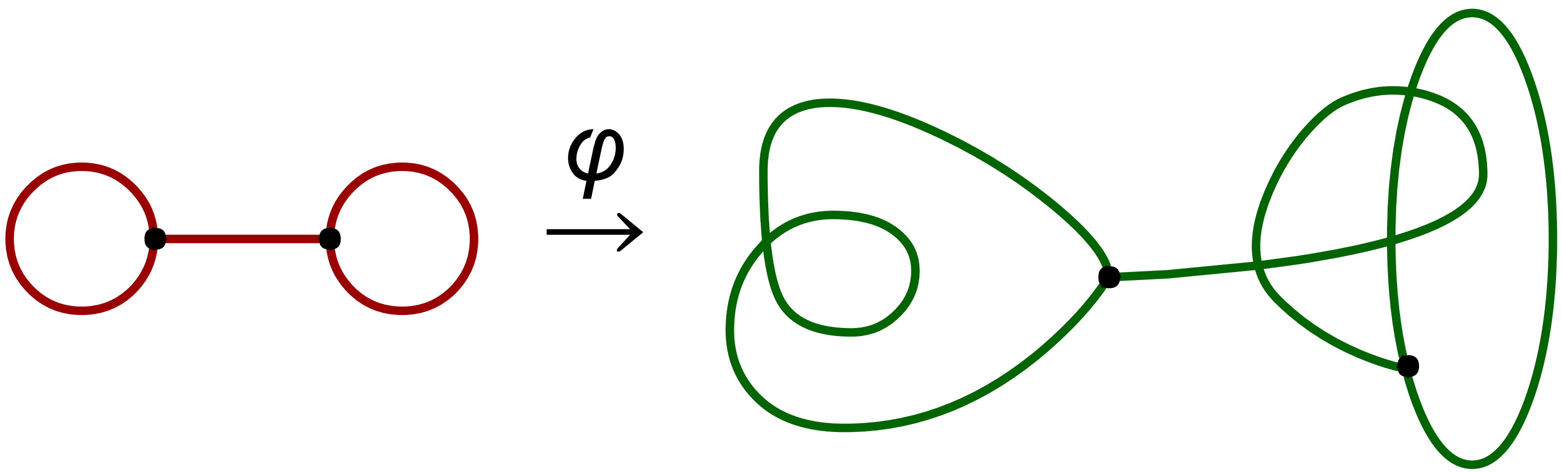
φ is identifiable.

$\overset{def.}{\Leftrightarrow}$ $^{\forall}f, f'$: spatial embeddings from φ

$f \sim f'$

We call φ IP.

ex.



2.1 SATに関連すること

φ : projection of G

φ is almost identifiable.

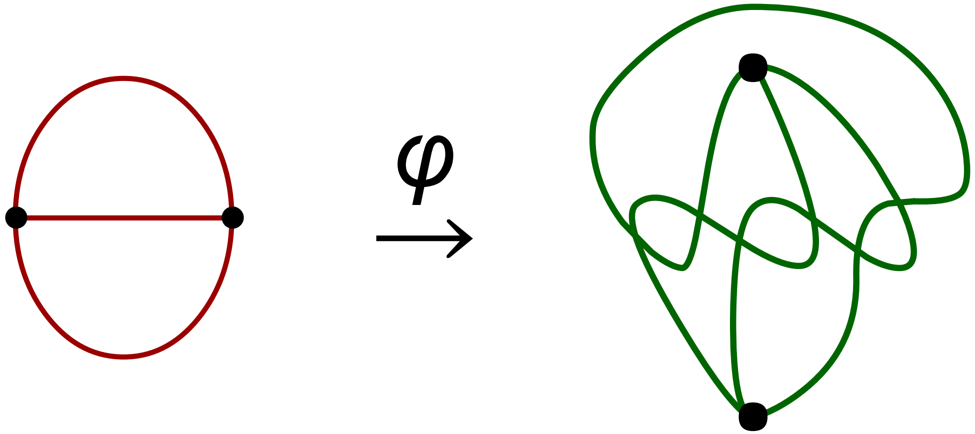
$\overset{def.}{\Leftrightarrow}$ $^{\forall}H \subsetneq G$: proper subgraph, $\varphi|_H$ is identifiable

We call φ AIP.

Hence, φ of G is identifiable. $\Rightarrow$ φ is almost identifiable.

φ of G is a SAT projection. $\Rightarrow$ φ is almost identifiable.

ex.



2.1 SATに関連すること

Proposition 1 [Huh-Taniyama, 2002]

Only planar graphs have identifiable projections.

Theorem 2 [Nikkuni, 2005]

φ : projection of G

φ is identifiable

$\Leftrightarrow$ $^{\forall}f$: spatial embedding obtained from φ,
f is trivial

2.1 SATに関連すること

Theorem 3 [Nikkuni, 2005]

G : planar graph which does not have a SAT embedding

φ : projection of G

φ is IP. $\Leftrightarrow$ φ is AIP.

SAT埋め込みをもたないグラフのIPの判定は,
真部分グラフのIPを見ればよい

2.2 知られていること

Question 1
What kind of planar graphs have
minimally knotted spatial embeddings ?

Ans. Every planar graph without vertices of degrees $\leqq 1$
（[Kawauchi, 1989], [Wu, 1993]）.

2.2 知られていること

Question 2
What kind of planar graphs have SAT embeddings ?

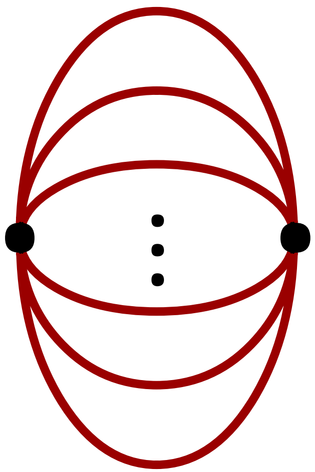
Partial Ans.

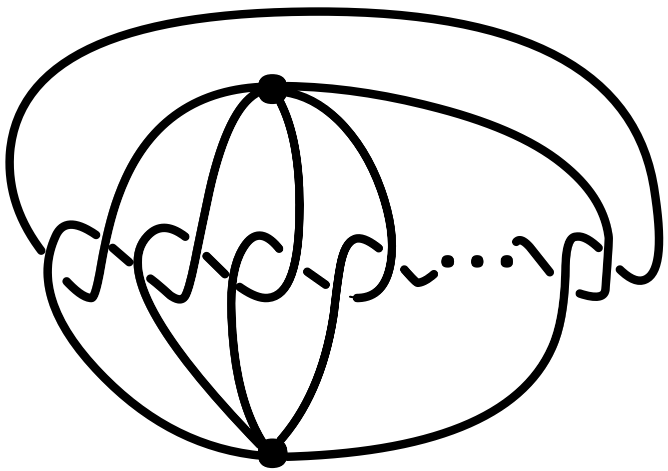
$^\exists G$: planar graph which has a SAT embedding

$^\exists G$: planar graph which does not have a SAT embedding

However it is not well known.

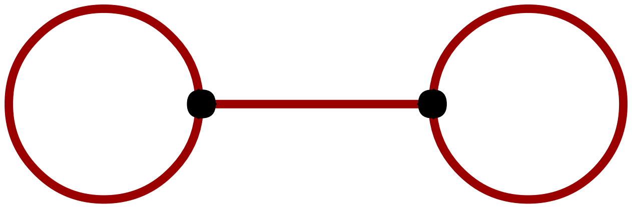
ex. θ_n -curve has SAT embeddings

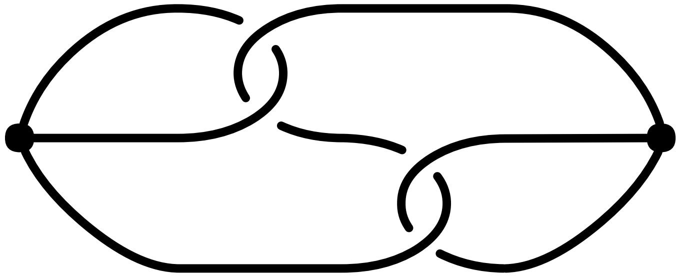




2.2 知られていること

ex.　　Handcuff graph has SAT embeddings.



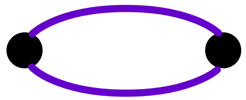


2.2 知られていること

Theorem 4 [Huh-Oh, 2003]

G : connected planar graph which does not have a cut vertex

G satisfies the followings

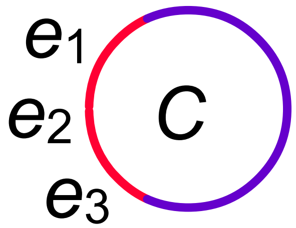
(1) G has no multiple edges. 

(2) $\forall e_1, e_2 \in E(G)$ s.t. $e_1 \cap e_2 = \emptyset$,

$\exists C_1, C_2$: disjoint cycles s.t. $e_1 \in E(C_1)$, $e_2 \in E(C_2)$

(3) $\forall e_1, e_2, e_3 \in E(G)$ s.t. $e_1 \cup e_2 \cup e_3$ is homeo. to a path,

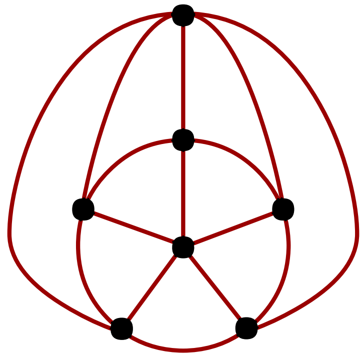
$\exists C$: cycle s.t. $e_1, e_2, e_3 \in E(C)$



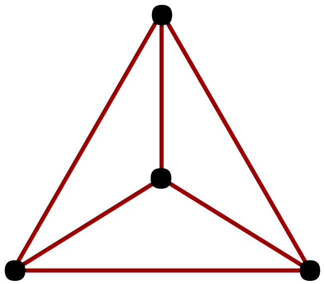
$\Rightarrow$ G has no strongly almost trivial embeddings.

2.2 知られていること

ex. graphs which does not have a SAT embedding

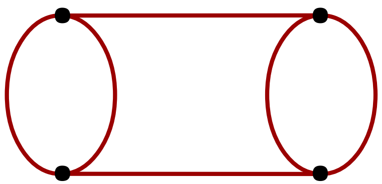


P_5 satisfies all assumptions of Thm.4



K_4 does not satisfy the assumption (2) of Thm. 4,
but it does not have a SAT embedding

[Huh-Oh, 2002]



Double-handcuff graph does not satisfy the assumptions (1) and (2),
but it does not have a SAT embedding

[Hanaki, preprint]

2.2 知られていること

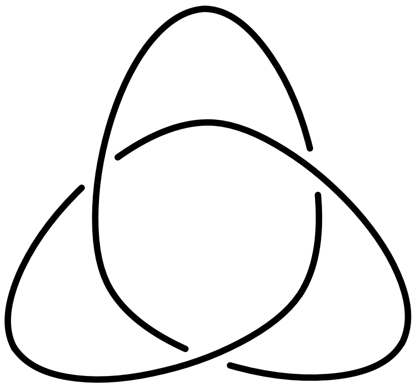
Question 3

Does $G = S^1{}_1 \amalg S^1{}_2 \amalg \cdots \amalg S^1{}_n$ have SAT embeddings ?

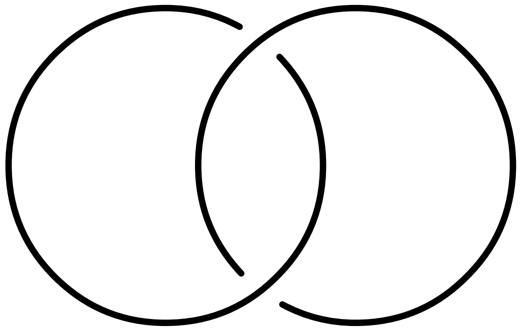
Ans. It has SAT embeddings if $n = 1,2$

It has no SAT embeddings if $n \geqq 3$

ex.



$n = 1$

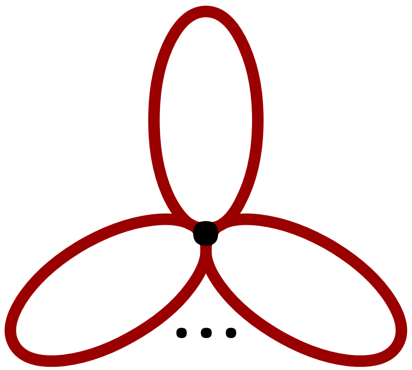


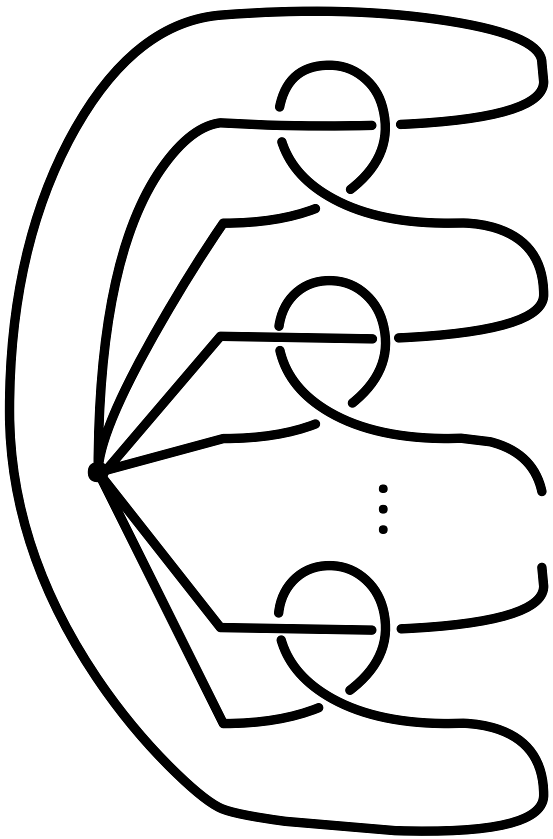
$n = 2$

3.1 わかったこと

Main Theorem 1 [Hanaki]

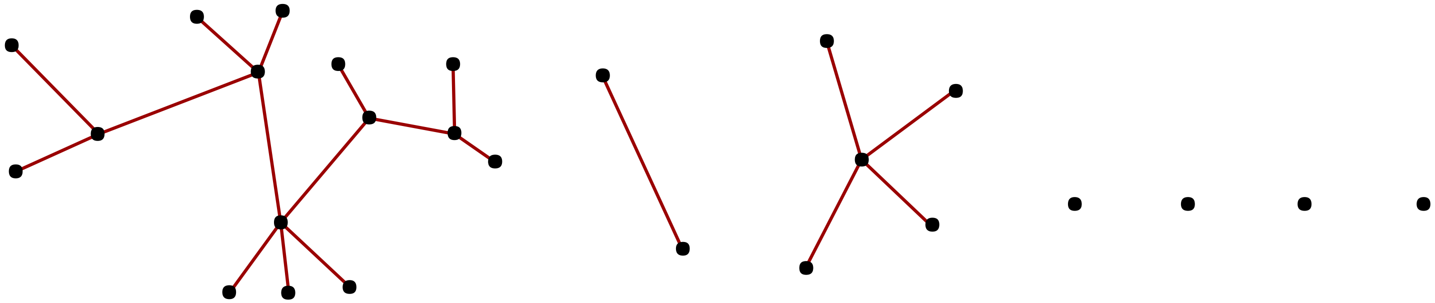
n-bouquet has SAT embeddings.



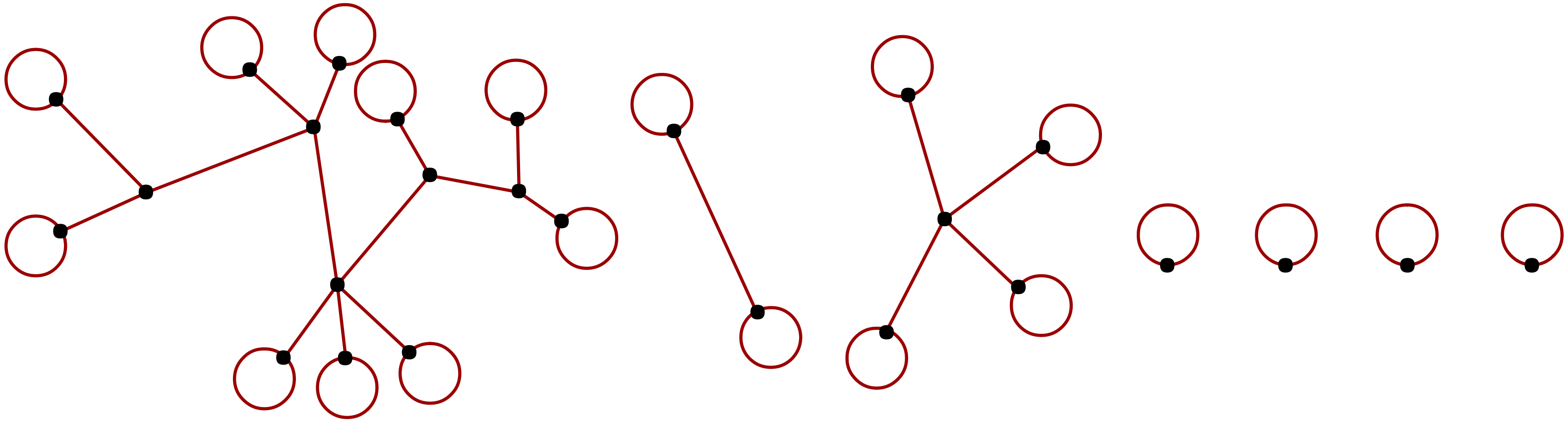


3.2 わかったこと2

F : forest



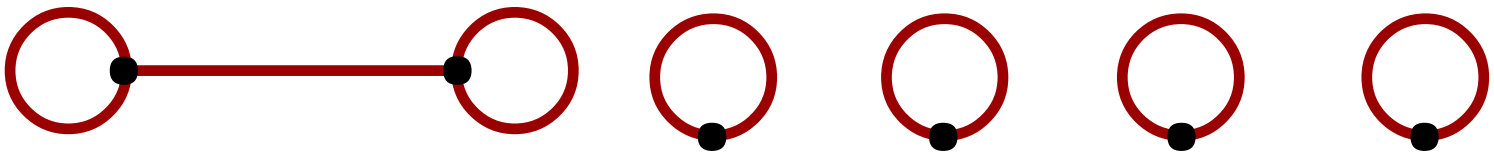
G_F : graph obtained from F

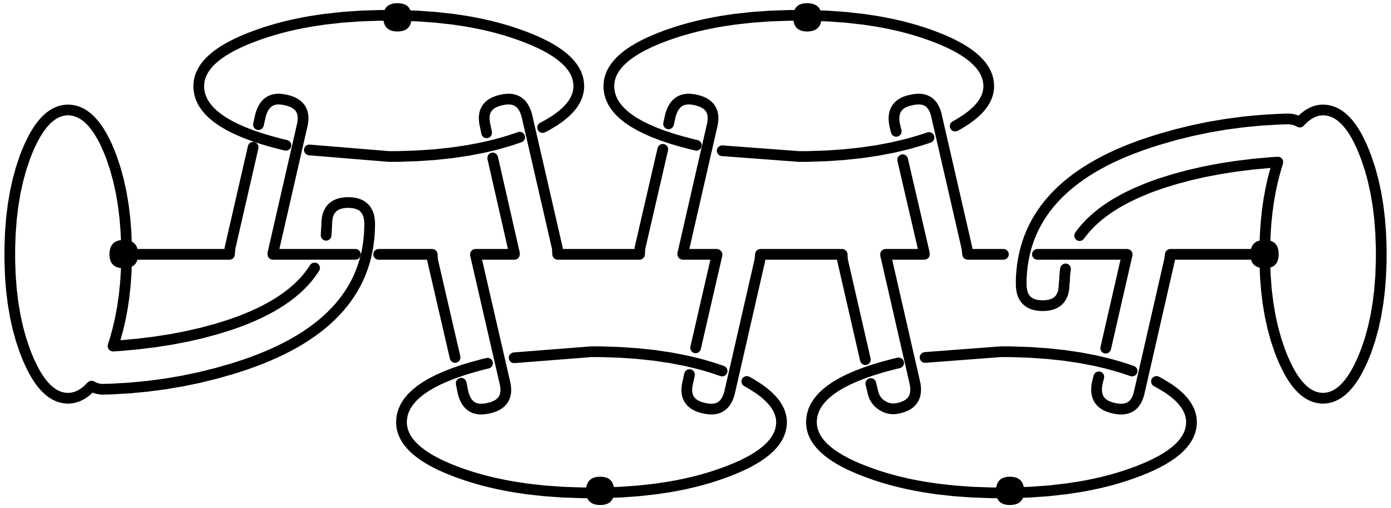


3.2 わかったこと 2

Main Theorem 2 [Hanaki]

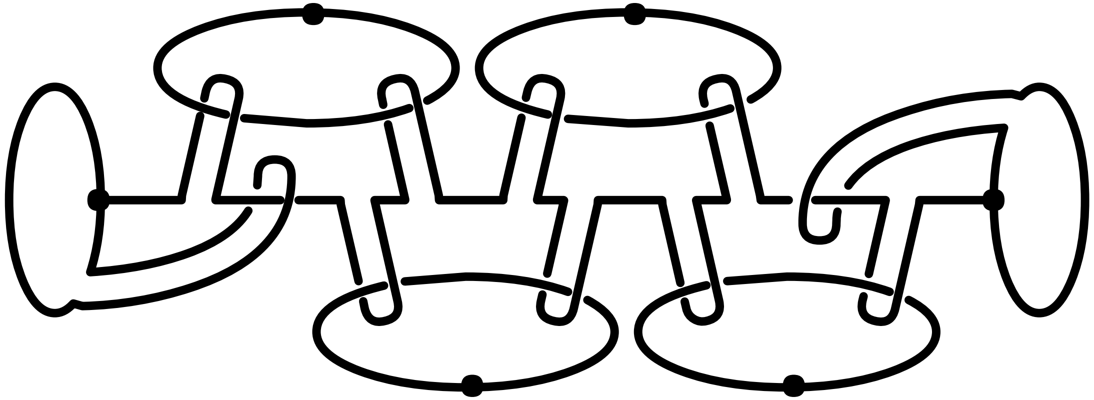
If $E(F) \neq \emptyset$, G_F has SAT embeddings.

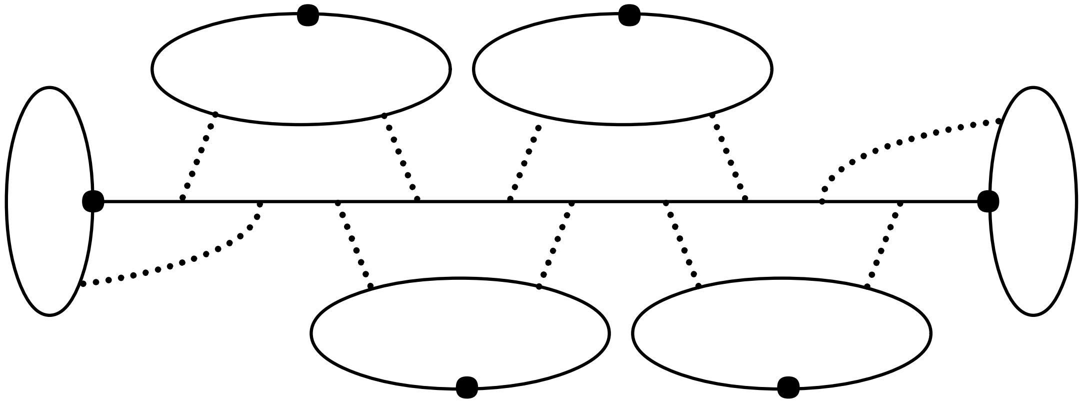




3.2 わかったこと２

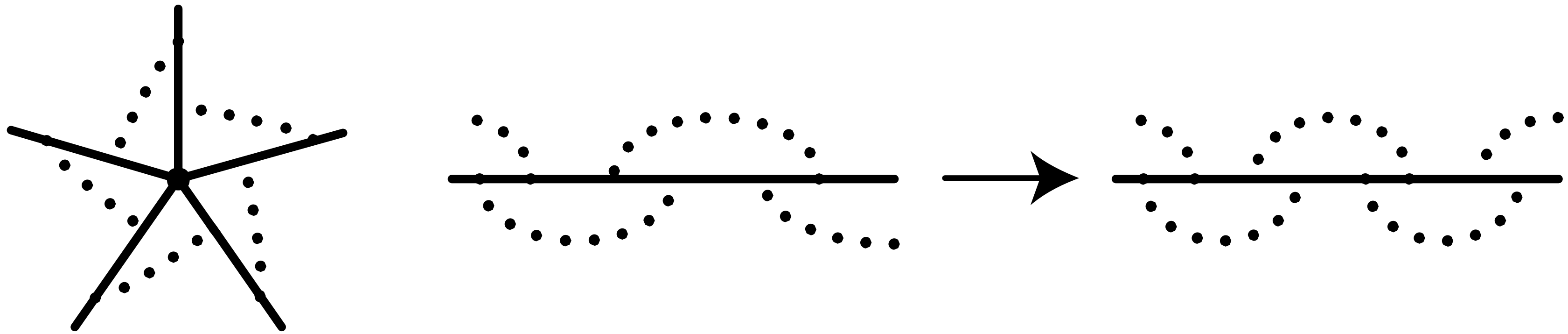
cord presentation

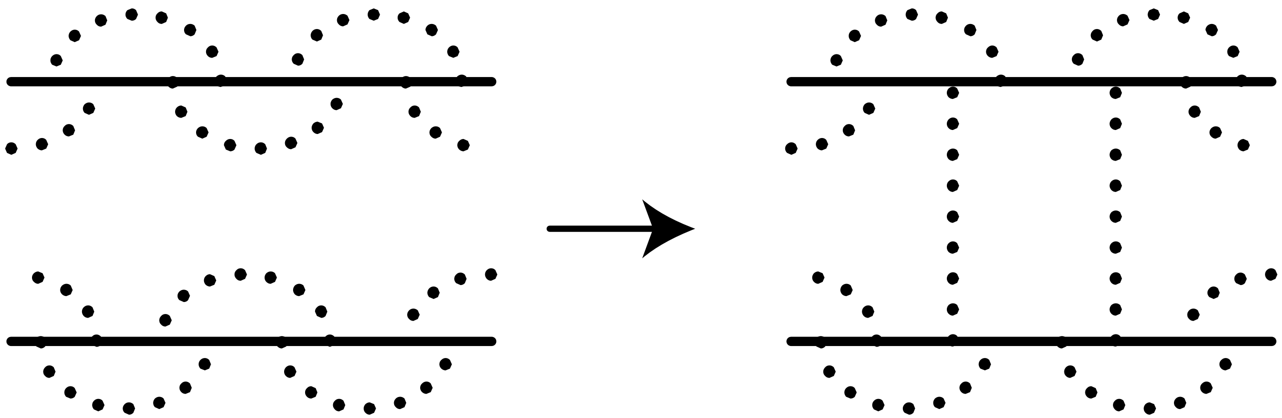




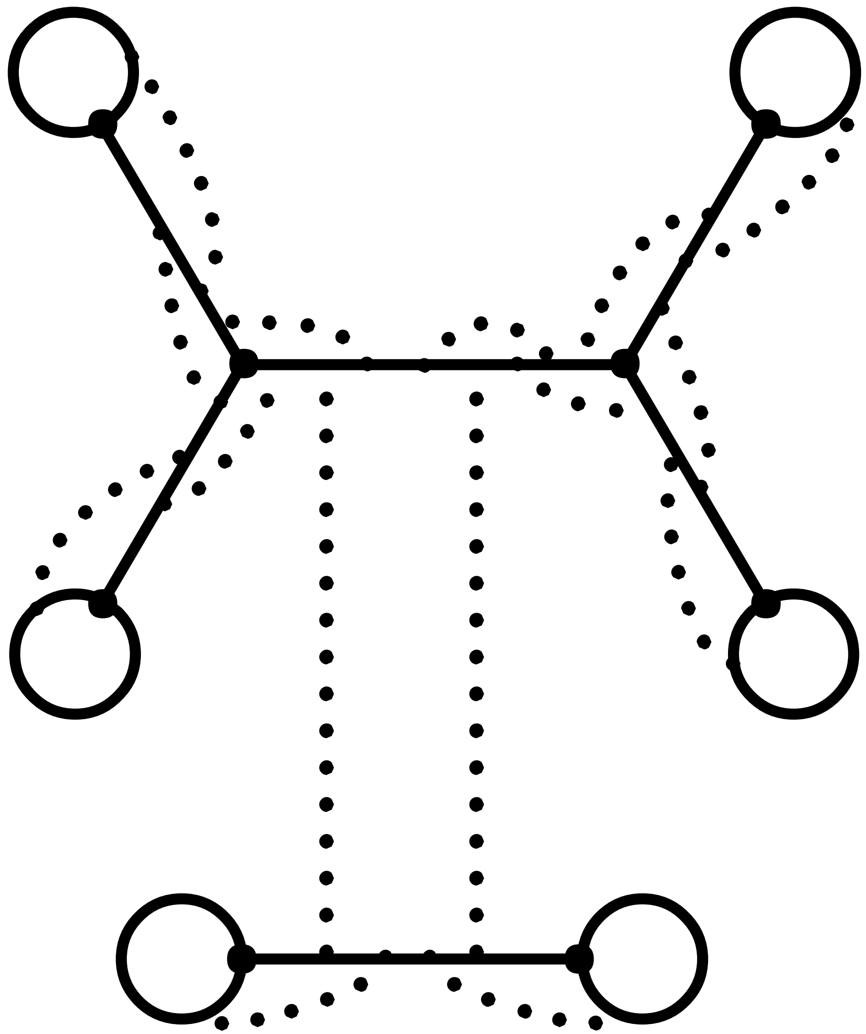
3.2 わかったこと２

SATの構成方法





3.2 わかったこと２



3.3 わかったこと3

Main Theorem 3 [Hanaki]

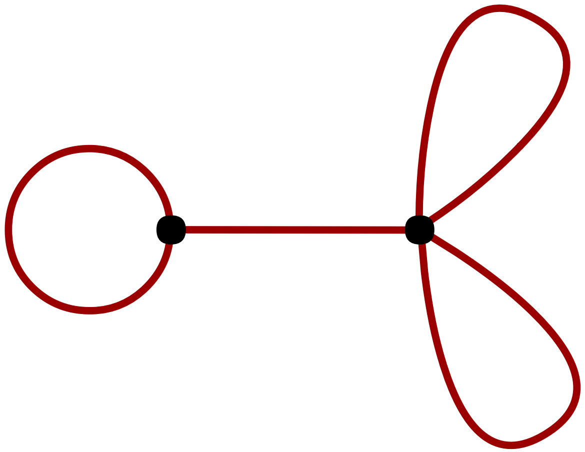
G : conn. graph s.t. G is not homeo. to handcuff graph

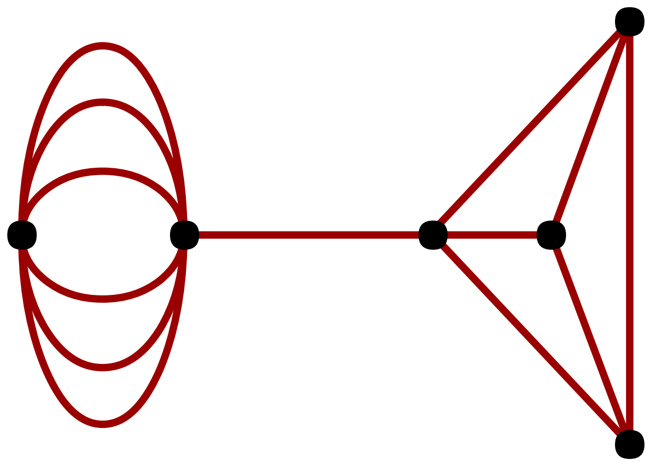
$\exists ! e \in E(G)$ s.t. e is a cut edge

H_1, H_2 : conn. comp. of $G - e$

H_1, H_2 has no cut edges and has cycles

⇒ G has no SAT embeddings

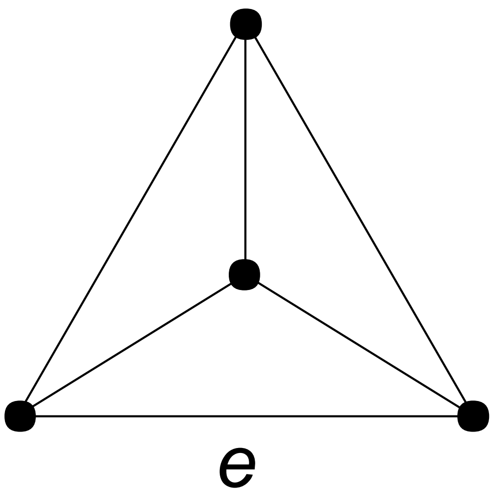




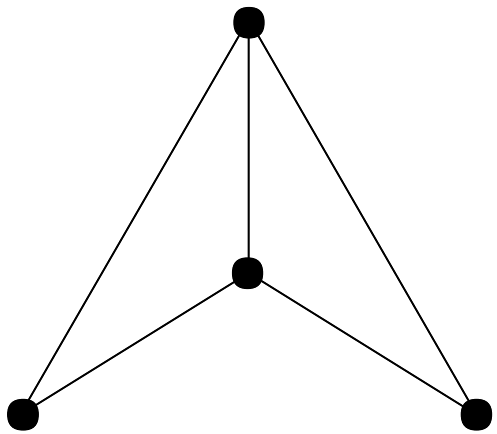
4.1 グラフマイナーについて

Edge deletion ($G-e$) is
the process of removing only an edge.

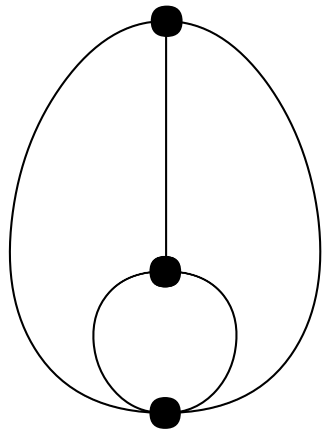
Edge contraction (G/e) is the process of removing an edge
and combining its two endvertices into a single vertex
where e is not loop



K_4



K_4-e



K_4/e

4.1 グラフマイナーについて

H is a minor of G ($H <_m G$)

$\overset{def.}{\Leftrightarrow}$ H can be obtained from G by contracting edges, deleting edges, and deleting isolated vertices

A property $\mathcal{P}$ of a graph is inherited by minors

$\overset{def.}{\Leftrightarrow}$ If G has $\mathcal{P}$, ${}^\forall H <_m G$, H has $\mathcal{P}$

Robertson-Seymour's Minor Theorem

G has $\mathcal{P}$

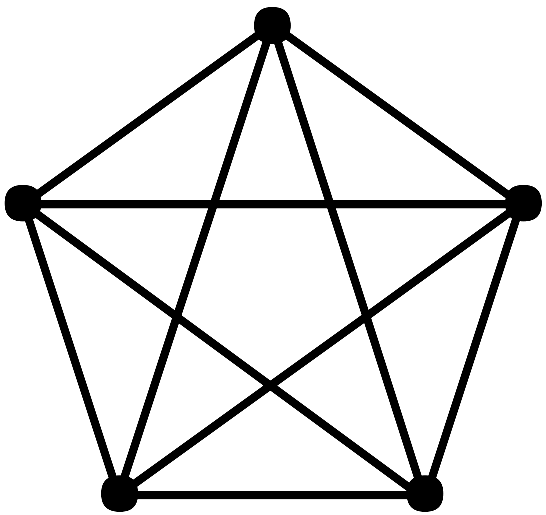
$\Leftrightarrow G$ does not contain $\underbrace{G_1, G_2, \cdots \text{ and } G_n}_{\text{finite}}$ as a minor

4.1 グラフマイナーについて

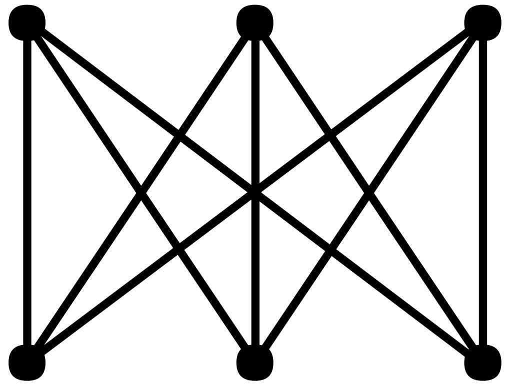
ex. Kuratowski Theorem

G is planar

⇔ G does not contain K_5, $K_{3,3}$ as a minor



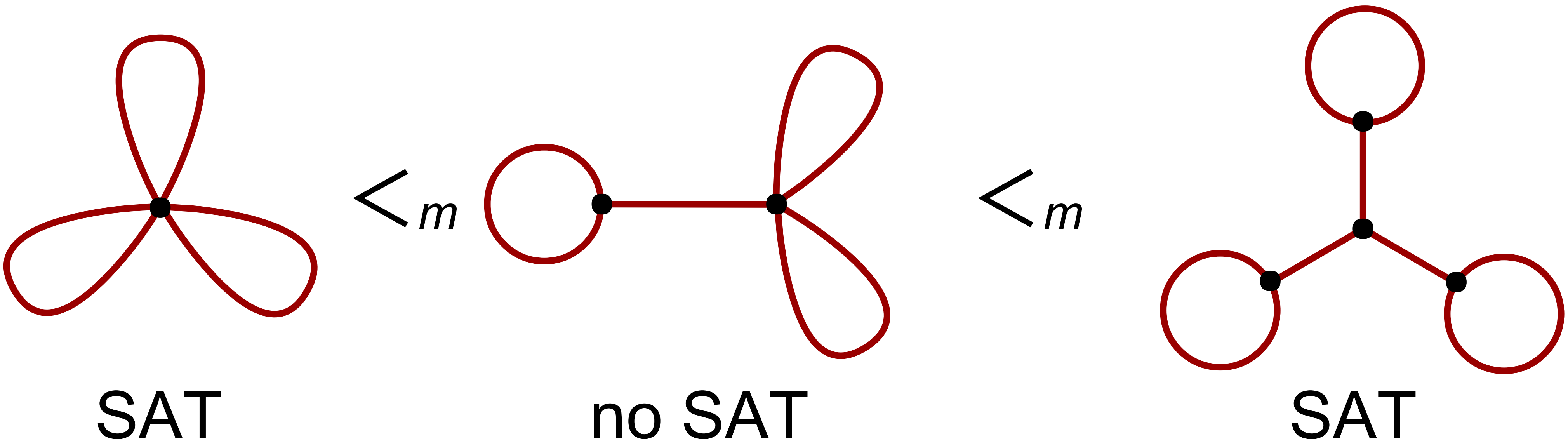
K_5



$K_{3,3}$

4.1 グラフマイナーについて

Remark
A property that a graph has (no) SAT embeddings
is not inherited by minors.



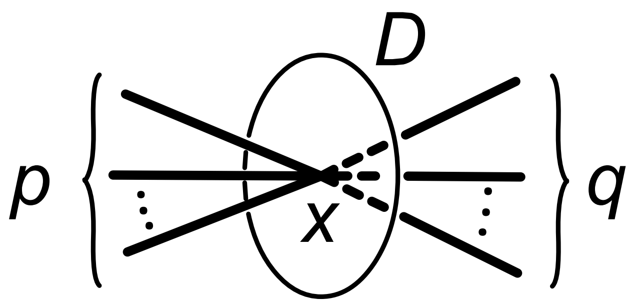
4.2 証明の一部のスケッチ

$G \subset S^3$ is irreducible

$\overset{def.}{\Leftrightarrow}$ $\nexists S \subset S^3$: 2-sphere s.t. G intersects both comp. of $S^3 - S$ and $|G \cap S| \leqq 1$

$G \subset S^3$: spatial graph
$D \subset S^3$: disk
D is good for G

$\overset{def.}{\Leftrightarrow}$ $\partial D \subset G$
int$D \cap G$ contains at most finitely many points
$x \in$ int$D \cap G$,
a neighbourhood of x looks like
where $p, q \in \boldsymbol{N}$

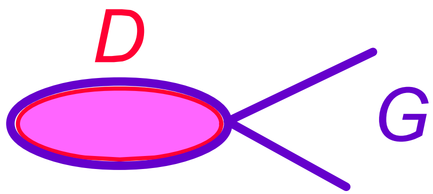


4.2 証明の一部のスケッチ

Theorem [Taniyama, 2002]

$G \subset S^3$: spatial graph

$D \subset S^3$: disk s.t. D is good for G



$\mathrm{int}D \cap G = \emptyset$ or $\partial D \cap \mathrm{cl}(G - \partial D)$ is not singleton

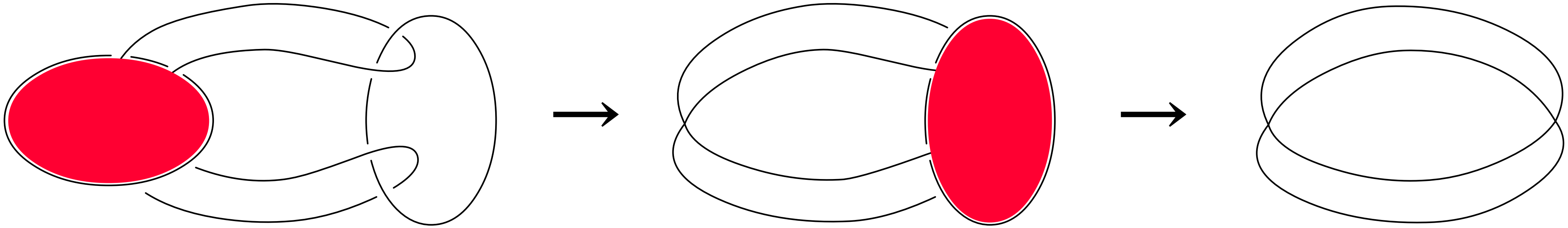
where cl denotes the closure

$G' \subset S^3$: spatial graph obtained from G by contracting D to a point

G' is irreducible $\Rightarrow$ G is irreducible

4.2 証明の一部のスケッチ

ex.



4.2 証明の一部のスケッチ

ex.