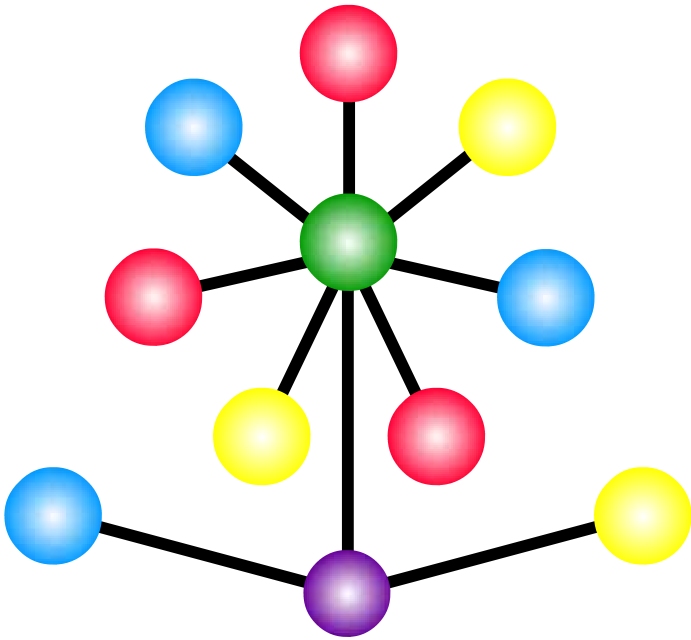
Pseudo diagrams of knots, links and spatial graphs



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1.1 Definitions

G : finite graph

f is a spatial embedding of G

$\overset{def.}{\Longleftrightarrow}$ $f : G \rightarrow \mathbf{S}^3$: embedding

We call $f(G)$ a spatial graph. In particular,
$f(G)$ is called a knot if G is homeomorphic to a circle and
a link if G is homeomorphic to disjoint circles.

$\mathcal{G}$, $\mathcal{G}'$: spatial graphs of G

$\mathcal{G}$ and $\mathcal{G}'$ are equivalent ($\mathcal{G} \sim \mathcal{G}'$)

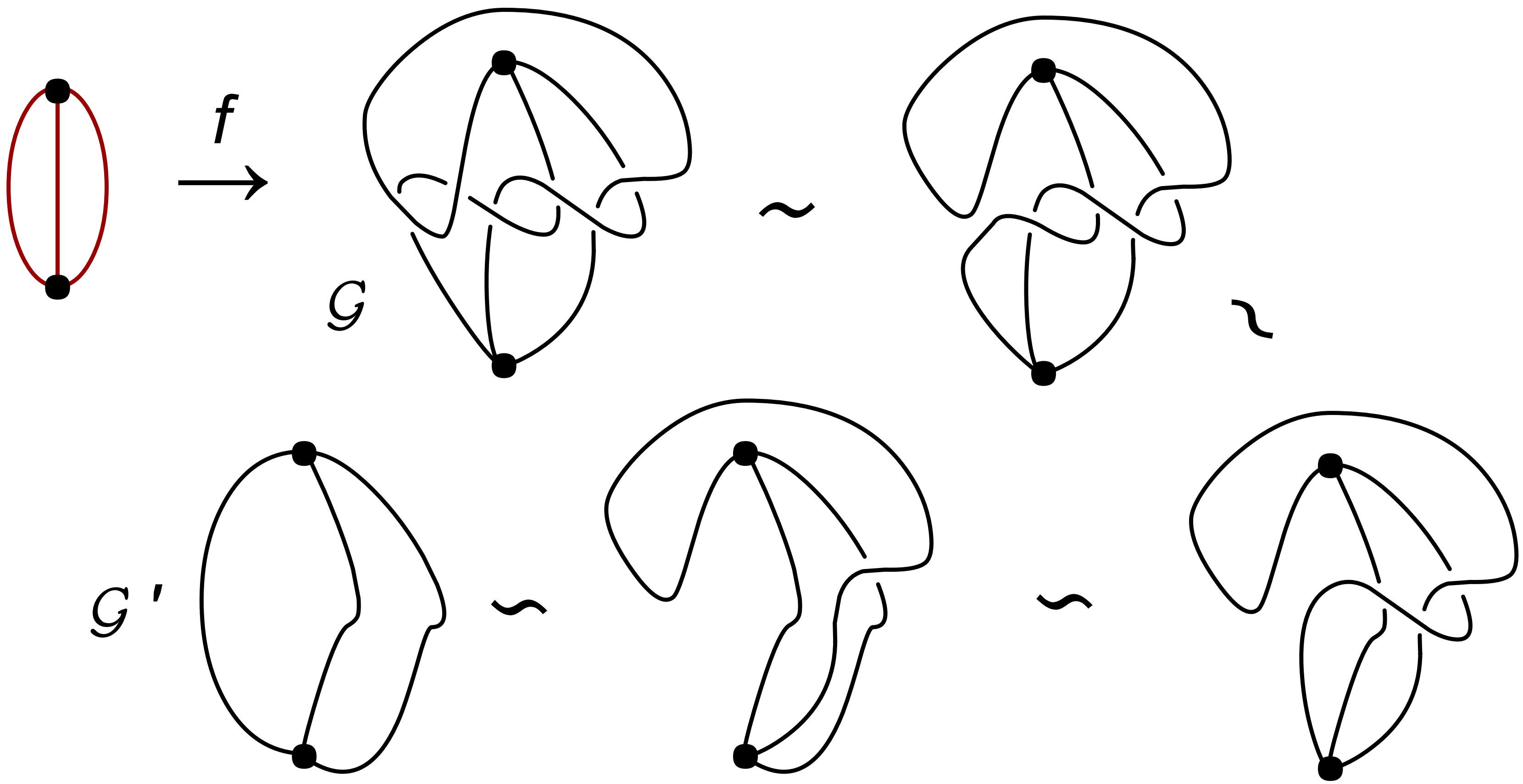
$\overset{def.}{\Longleftrightarrow}$ $^\exists h : S^3 \rightarrow S^3$: orientation preserving self-homeomorphism
s.t. $h(\mathcal{G}) = \mathcal{G}'$

1.1 Definitions

$\mathcal{G}$ is trivial (or unknotted)

$\overset{def.}{\Leftrightarrow}$ $^{\exists}\mathcal{G}' \sim \mathcal{G}$ s.t. $\mathcal{G}' \subset \mathbf{S}^2 \subset \mathbf{S}^3$

ex.



1.1 Definitions

G is planar

$\overset{def.}{\Longleftrightarrow}$ ${}^{\exists}f : G \to \mathbf{S}^2$: embedding

Hence

G has a trivial spatial graph. $\Leftrightarrow$ G is planar.

We consider only planar graphs.

1.2 Definitions

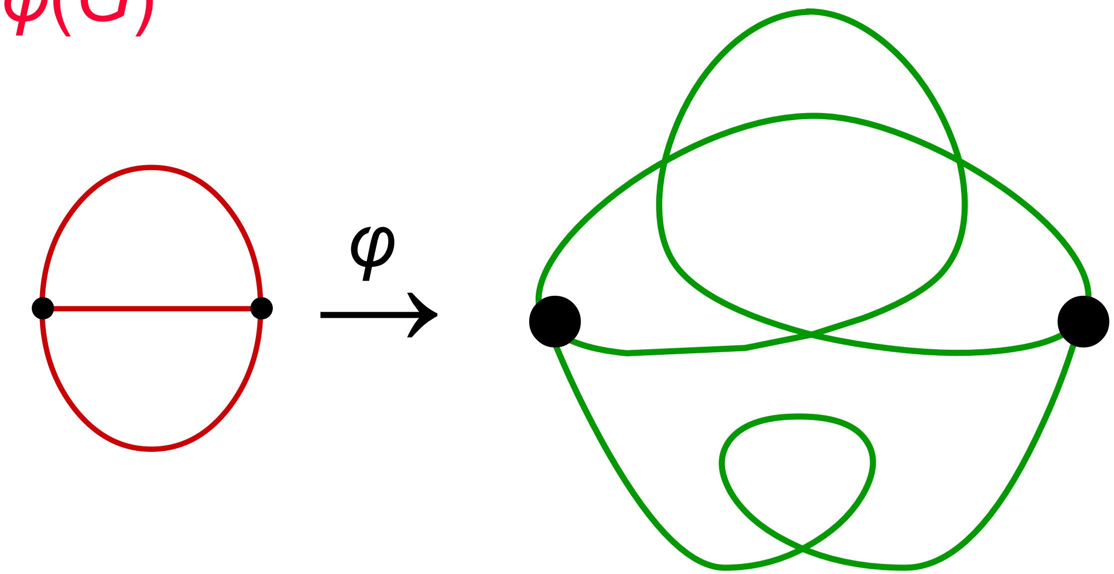
$\varphi : G \to \mathbf{S}^2$: continuous map

φ is a projection of G

$\overset{def.}{\Longleftrightarrow}$ multiple points of φ are only finitely many transversal double points away from the vertices

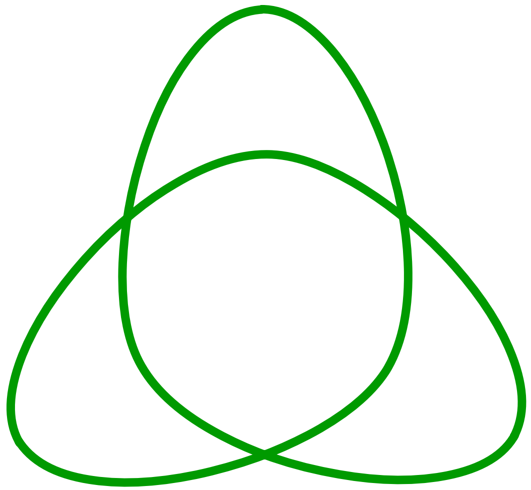
The image of a projection is also called a projection

and we denote it by $P = \varphi(G)$



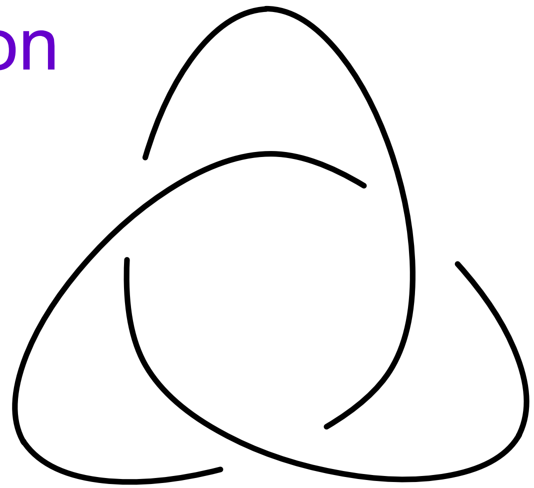
1.2 Definitions

A diagram *D* is a projection *P* with over/under information at each double point.
Then we say *D* is obtained from *P*.



A diagram *D* uniquely represents a spatial graph up to equivalence.

Then a double point with over/under information is called a crossing and a double point without over/under information is called a pre-crossing.



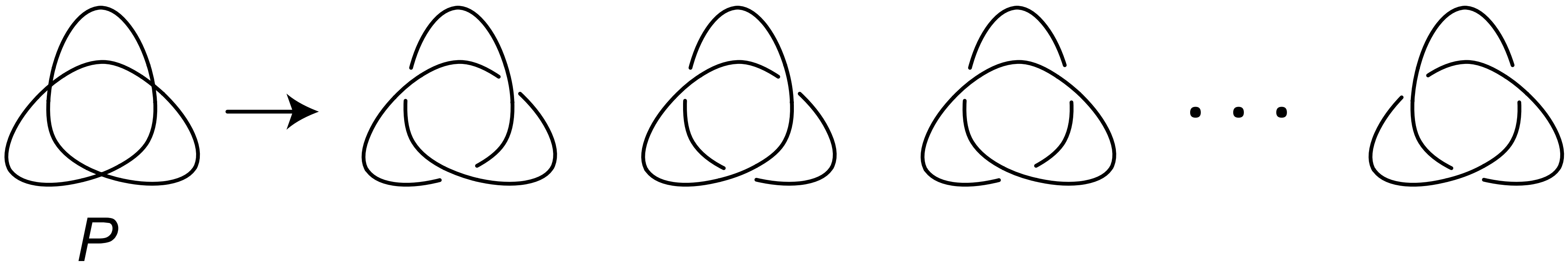
Thus a diagram has crossings and has no pre-crossings.

2.1 Motivation

Q. Can we determine from a projection P whether the original spatial graph $\mathcal{G}$ is trivial or knotted?

Ans. We cannot determine except some special cases.

ex.



Two diagrams representing a nontrivial knot and six diagrams representing a trivial knot are obtained from P.
Therefore we cannot determine.

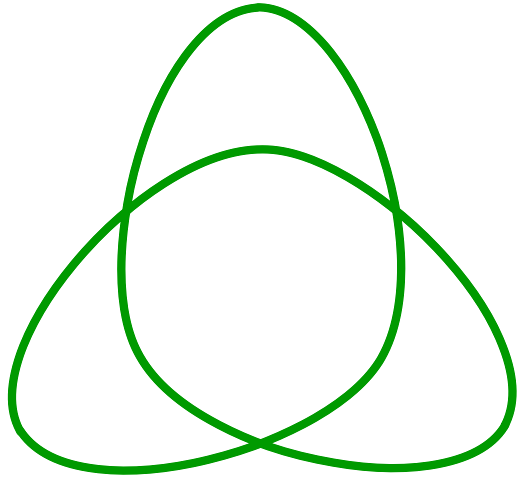
2.1 Motivation

Q. Which pre-crossings of a projection P and which over/under informations at them should we know in order to determine that the original spatial graph $\mathcal{G}$ is trivial or knotted?

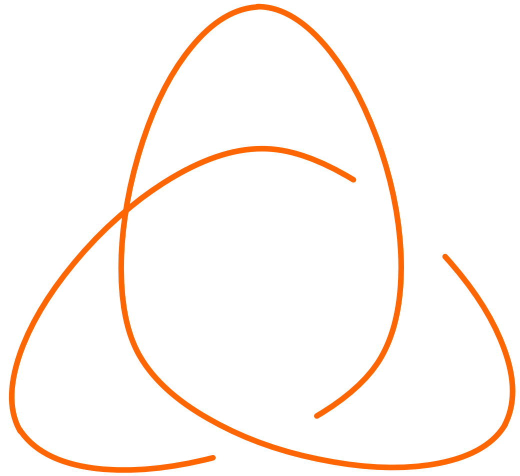
Now we introduce the notion of a pseudo diagram.

2.2 Definitions

A pseudo diagram *Q* is a projection *P* with over/under information at some pre-crossings. Then we say *Q* is obtained from *P*.



Thus a pseudo diagram *Q* has crossings and pre-crossings. Here Q possibly has no crossings or has no pre-crossings. Namely, Q is possibly a projection or a diagram.

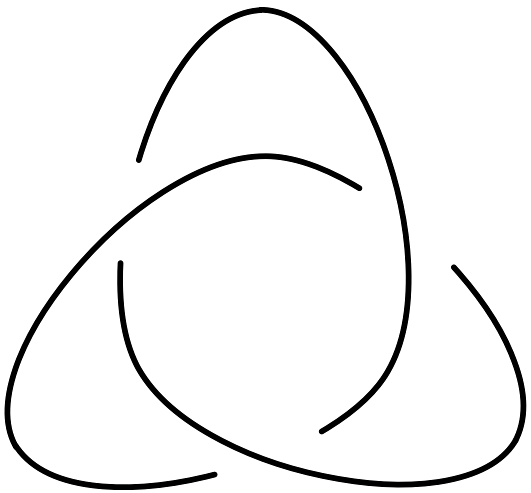


2.2 Definitions

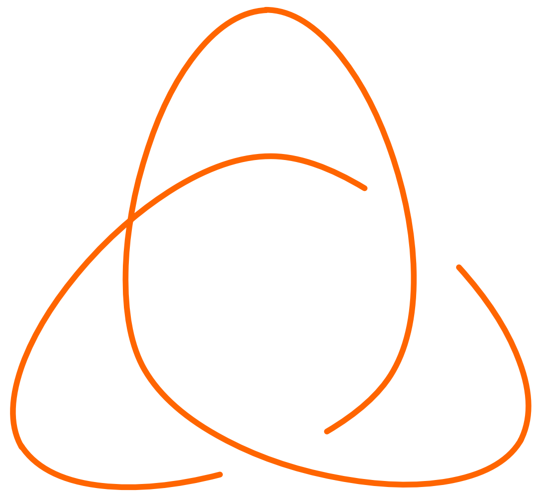
A pseudo diagram Q' is obtained from a pseudo diagram Q.

def. $\Leftrightarrow$ Each crossing of Q has the same over/under information as Q'.

ex.



Q'



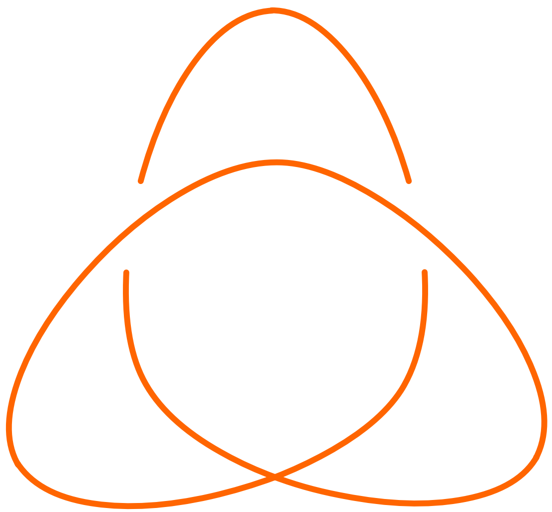
Q

2.3 Definitions

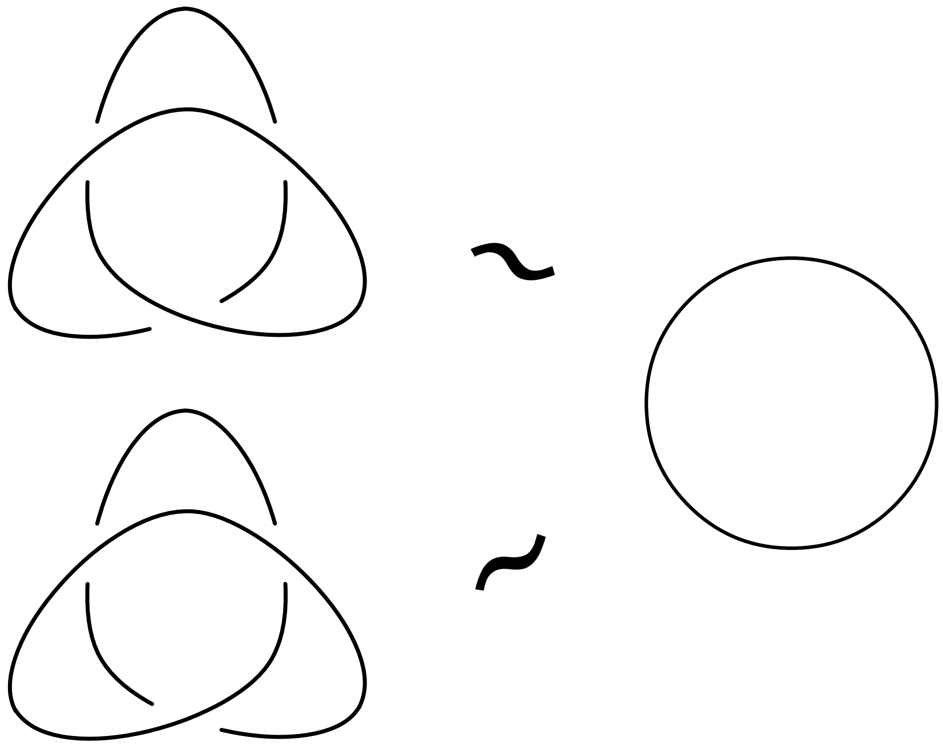
pseudo diagram Q is trivial.

def. $\Leftrightarrow$ $\forall D$: diagram obtained from Q
D represents a trivial spatial graph.

ex.



Q : trivial pseudo diagram



2.3 Definitions

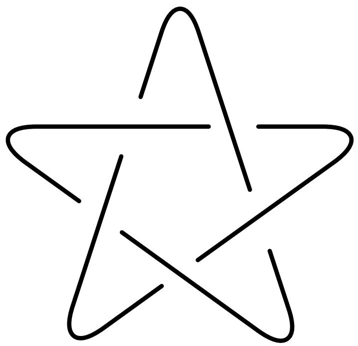
pseudo digaram Q is knotted.

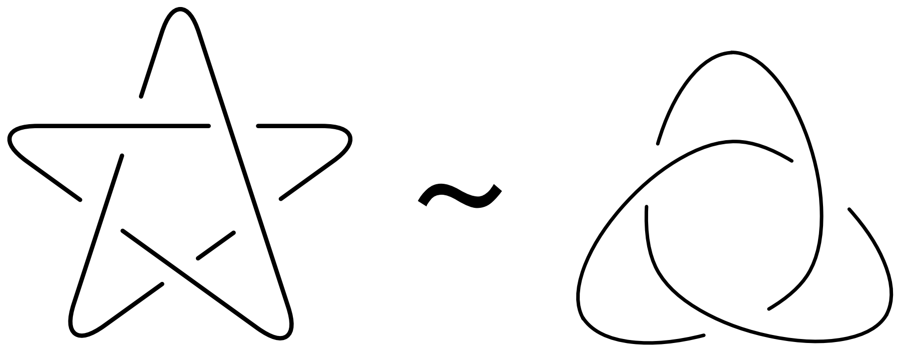
def. $\forall D$: diagram obtained from Q

$\Leftrightarrow$ D represents a nontrivial spatial graph.

ex.

Q : knotted pseudo diagram

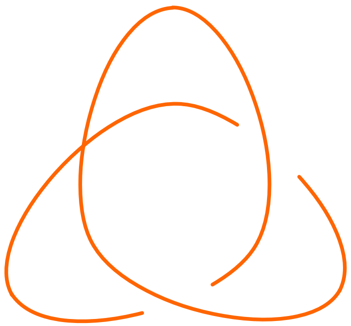




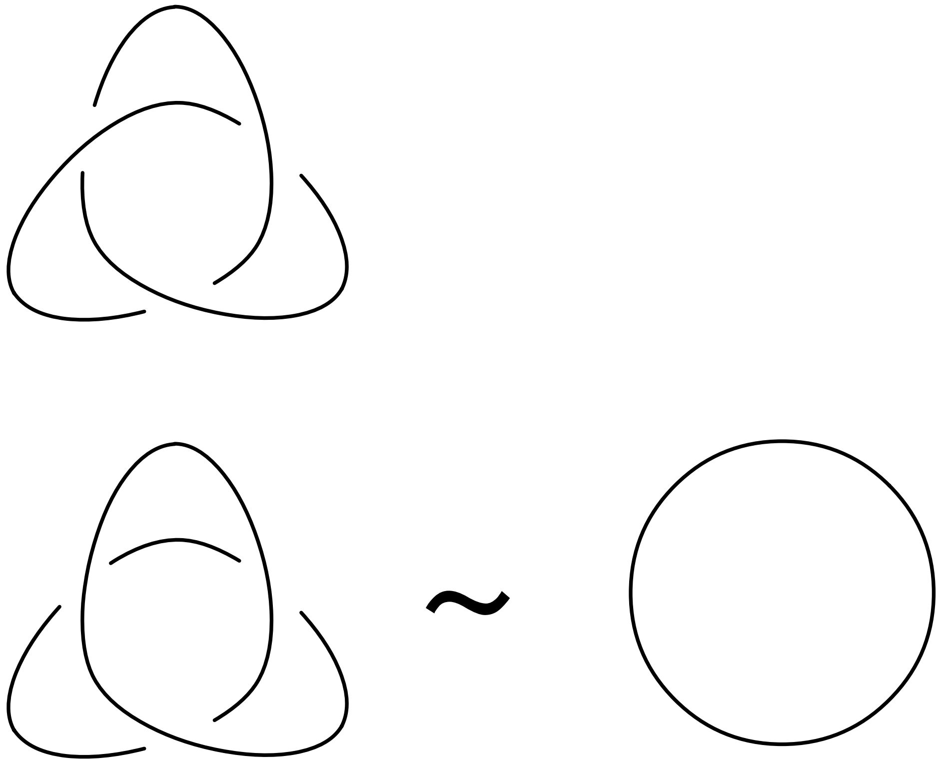
2.3 Definitions

There exists a pseudo diagram
which is neither trivial nor knotted.

ex.



Q



2.3 Definitions

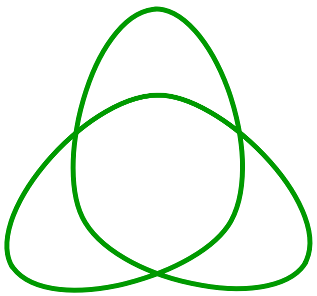
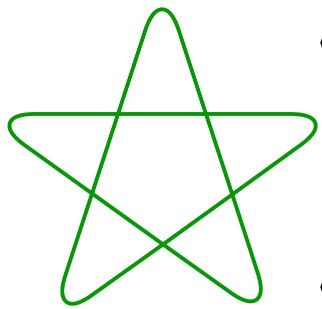
P : projection

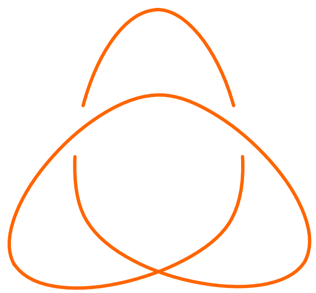
$$tr(P) := \min \left\{ c(Q) \,\middle|\, Q : \text{trivial pseudo diagram obtained from } P \right\}$$

where $c(Q)$: the cardinality of the set of crossings of Q

We call $tr(P)$ the trivializing number of P.

ex.

$tr\Big($  $\Big) = 2$ $\qquad$ $tr\Big($  $\Big) = 4$



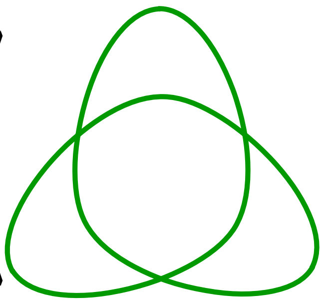
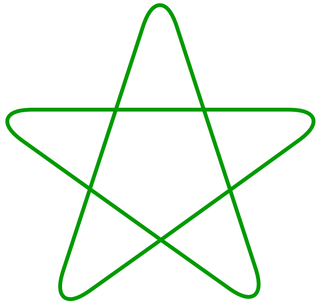
2.3 Definitions

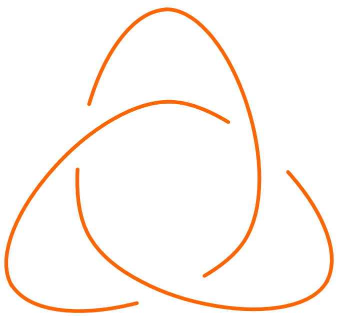
P : projection

$$kn(P) := \min \left\{ c(Q) \;\middle|\; \begin{array}{l} Q : \text{knotted pseudo diagram} \\ \quad\text{obtained from } P \end{array} \right\}$$

We call $kn(P)$ the knotting number of P.

ex.

$kn\Big($  $\Big) = 3$ $\qquad$ $kn\Big($  $\Big) = 4$



2.4 Remarks

$\forall G$: planar graph
$\exists P$: projection of G with $kn(P)=\infty$ i.e. $tr(P)=0$

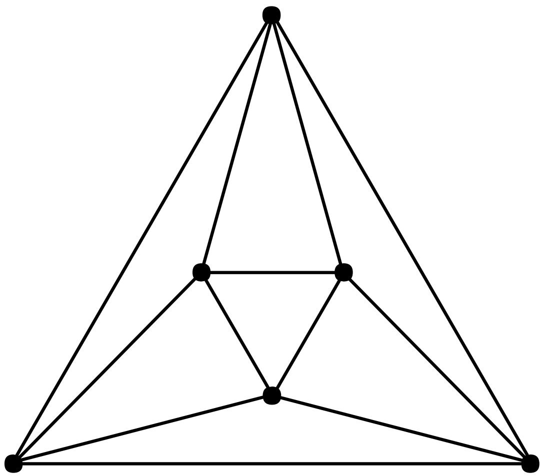
ex. $\varphi : G \to \mathbf{S}^2$: embedding
$P=\varphi(G)$

2.4 Remarks

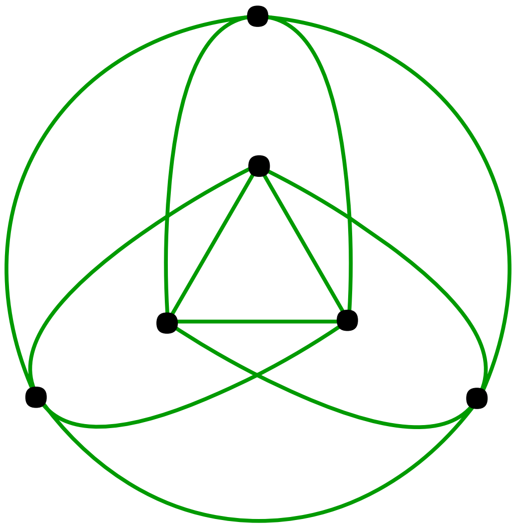
∃ G : planar graph

∃ P : projection of G with $tr(P)=\infty$ i.e. $kn(P)=0$

ex. [Taniyama, 1995]



G



P

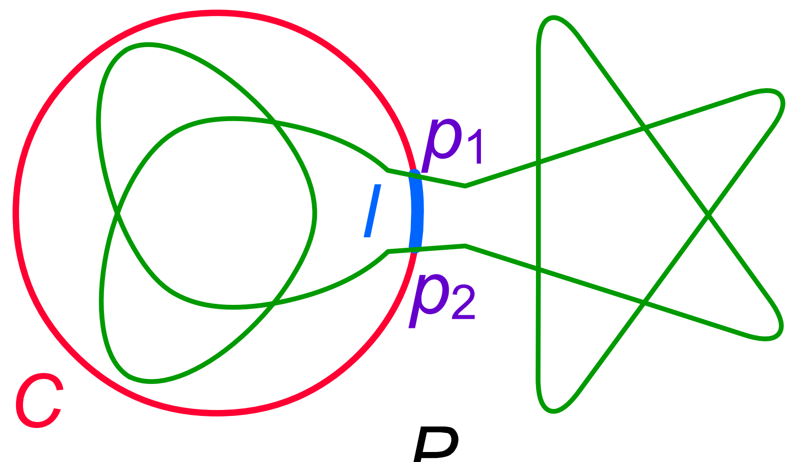
Any diagram obtained from P contains a diagram of a Hopf link.

3.1 Proposition

P : projection of $G = \mathbf{S}^1$

$C \subset \mathbf{S}^2$ is a decomposing circle of P

$\overset{def.}{\Leftrightarrow}$ $P \cap C = \{$ two transversal double points $\}$

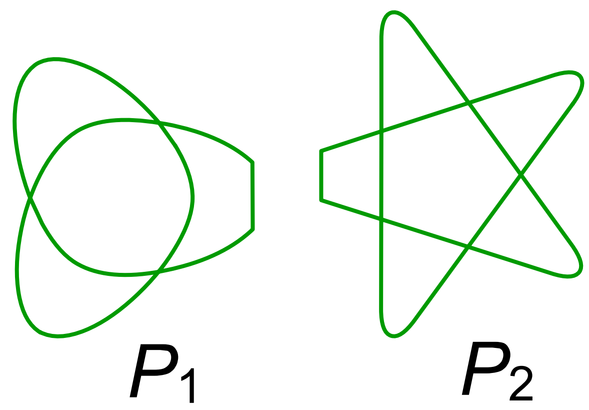


Here we put $\{ p_1, p_2 \} = P \cap C$

B_1, B_2 : disks s.t. $B_1 \cup B_2 = \mathbf{S}^2$, $B_1 \cap B_2 = C$

l : arc on C joining p_1 and p_2

$P_1 = (P \cap B_1) \cup l$, $P_2 = (P \cap B_2) \cup l$



Proposition 1

$tr(P) = tr(P_1) + tr(P_2)$

$kn(P) = \min \{ kn(P_1), kn(P_2) \}$

3.2 Trivializing number

Theorem 1

P : projection of $\mathbf{S}^1$ $\Rightarrow$ $tr(P)$ is always even.

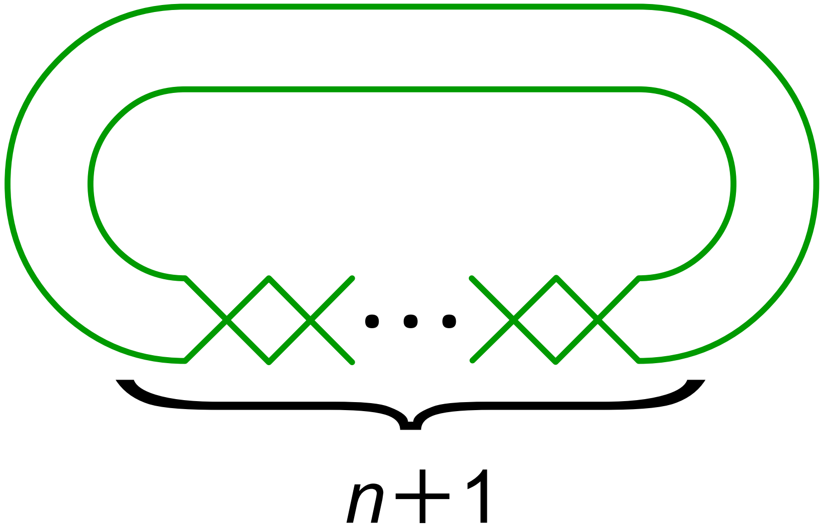
Proposition 2

$\forall n$: even positive number

$\exists P$: projection of S^1 with $tr(P)=n$

ex.

P:



$tr(P)=n$

3.2 Trivializing number

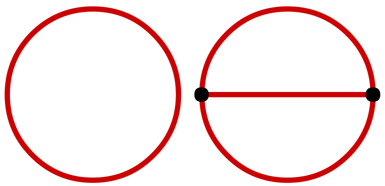
Corollary 1

P : projection of $\mathbf{S}^1 \amalg \mathbf{S}^1 \amalg \cdots \amalg \mathbf{S}^1 \Rightarrow$ $tr(P)$ is always even.

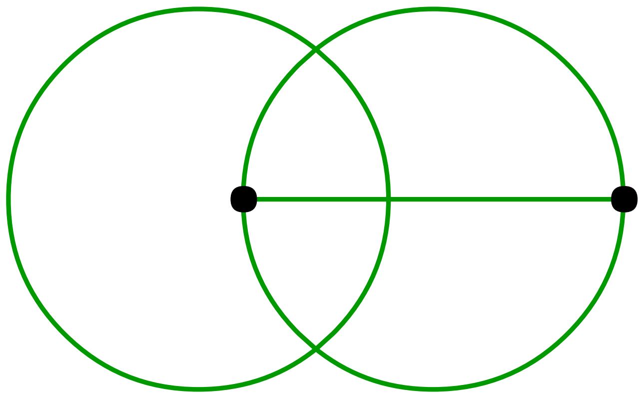
Proposition 3

G : planar graph
P : projection of G $\Rightarrow tr(P) \neq 1$

Remark

$G=$ 

$\exists P$: projection of G with $tr(P)=3$



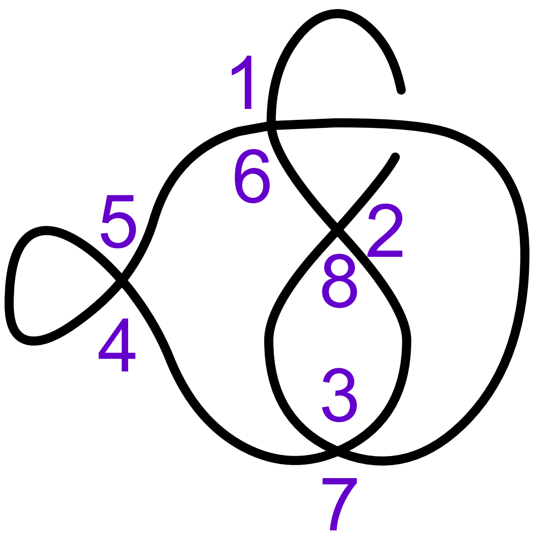
3.3 Trivializing number & Chord diagram

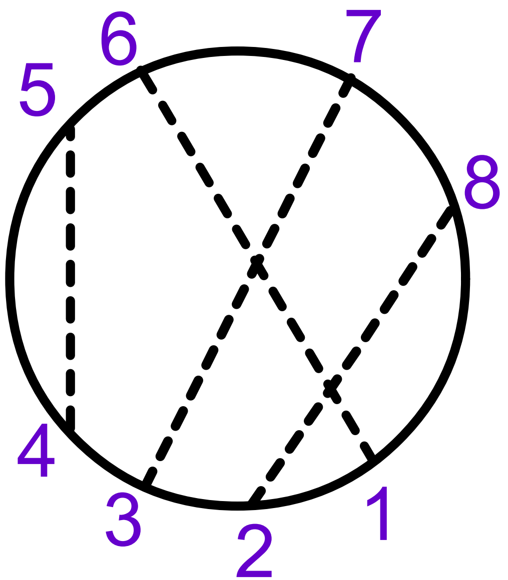
Q : pseudo diagram of S^1 with n pre-crossings

CD_Q is a chord diagram of Q

$\overset{def.}{\Leftrightarrow}$ CD_Q is a circle with n chords s.t. the preimage of each pre-crossing is connected by a chord

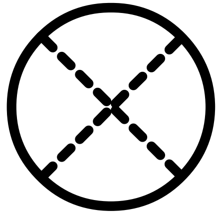
ex.





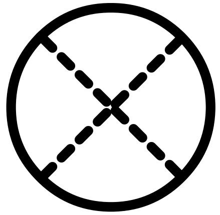
3.3 Trivializing number & Chord diagram

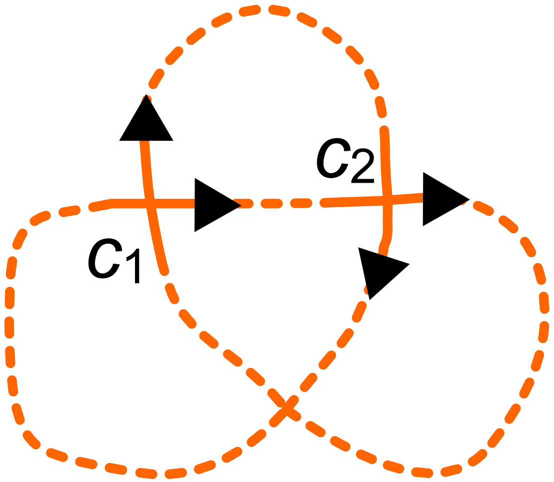
Lemma 1

Q : pseudo diagram s.t. CD_Q contains 

⇒ Q is not trivial.

Proof

Q' : a pseudo diagram obtained from Q with $CD_{Q'}$ = 



K_1: the knot represented by D_{++}
where ++ means that c_1 is + and c_2 is +

K_2: the knot represented by D_{+-}

K_3: the knot represented by D_{-+}

K_4: the knot represented by D_{--}

$$a_2(K_1) - a_2(K_2) - a_2(K_3) + a_2(K_4) = 1$$

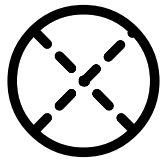
A diagram representing a nontrivial knot is obtained from Q.

3.3 Trivializing number & Chord diagram

Lemma 2

P : projection of $\mathbf{S}^1$

CD : sub-chord diagram of CD_P

s.t. CD does not contain 

⇒ ∃ Q : trivial pseudo diagram obtained from P

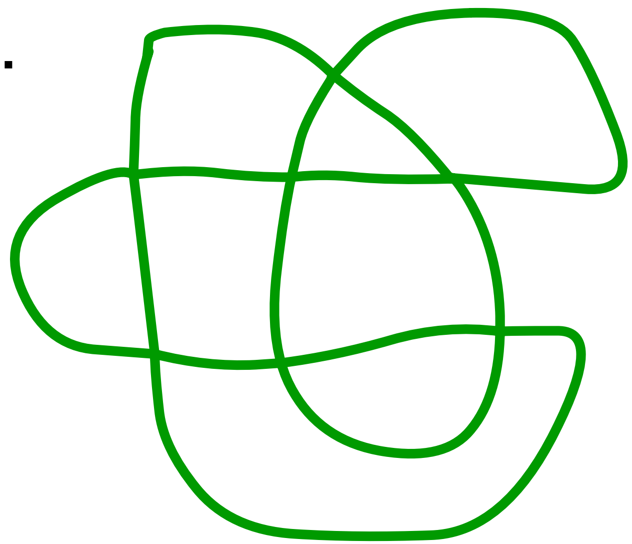
with $CD_Q = CD$

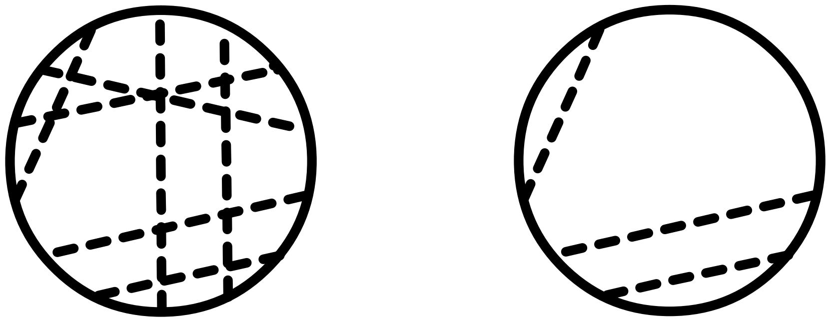
3.3 Trivializing number & Chord diagram

P : projection of $\mathbf{S}^1$

By using Lemma 1 and 2,
we can get the trivializing number of P from CD_P.

ex.





$\therefore tr(P)=4$

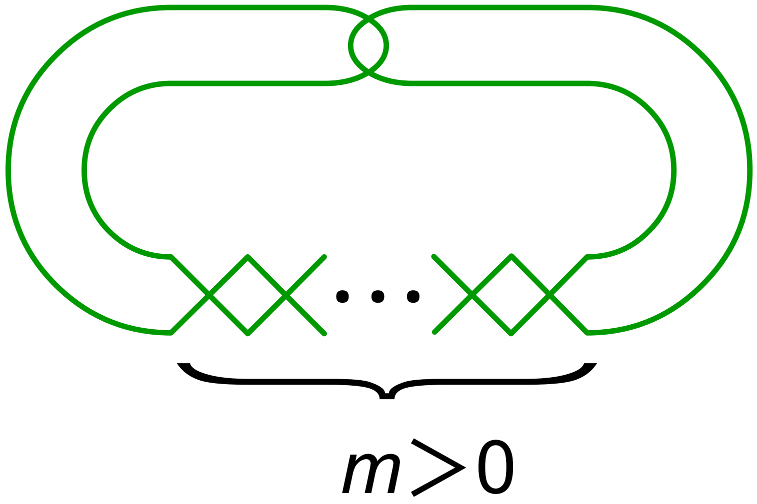
3.4 Theorems on Trivializing number

Theorem 2

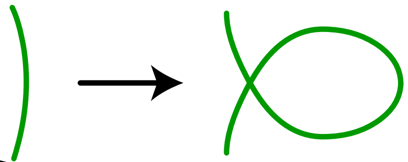
P : projection of $\mathbf{S}^1$

$tr(P) = 2$

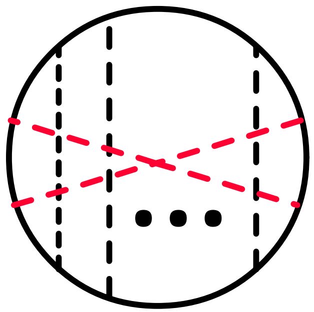
$\Leftrightarrow$ P is obtained from



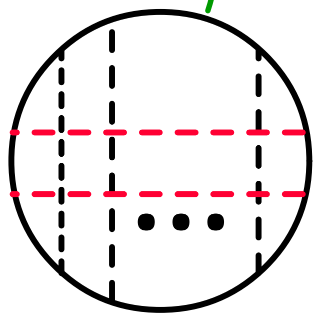
by a series of replacing a sub-arc of P as



Here CD_P is



if m is odd



if m is even

3.4 Theorems on Trivializing number

Theorem 3

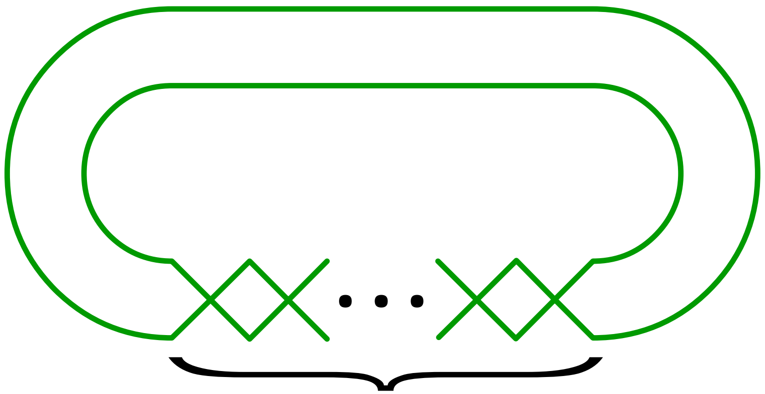
P : projection of $\mathbf{S}^1$

$\Rightarrow tr(P) \leqq p(P) - 1$

where $p(P)$: the cardinality of the set of pre-crossings of P

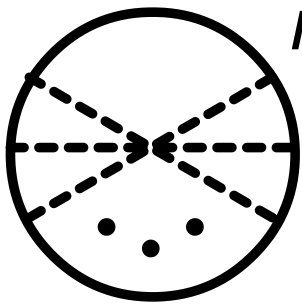
The equality holds

$\Leftrightarrow$ P is one of



m : odd

Here CD_P is



3.5 Knotting number

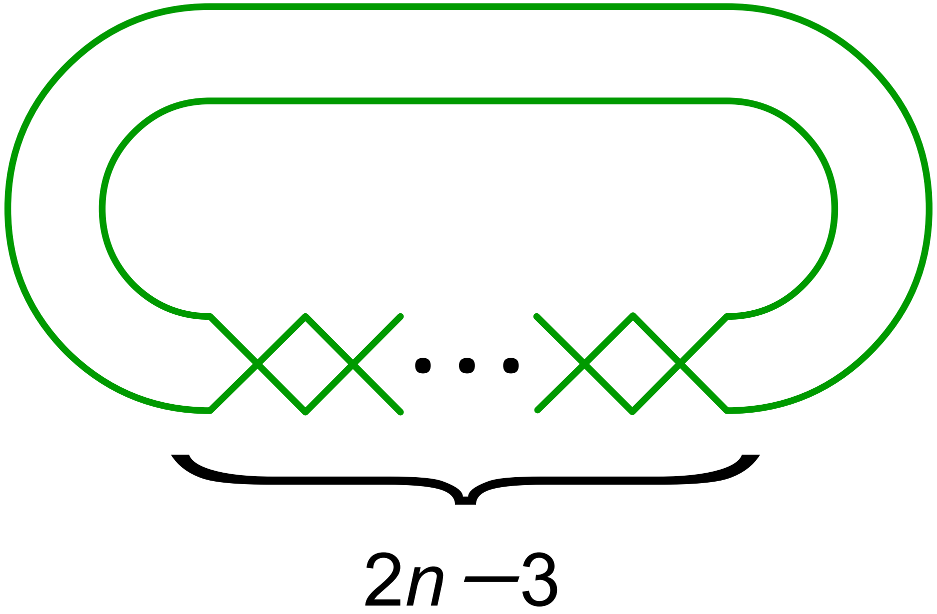
Proposition 4

$\nexists P$: projection of $\mathbf{S}^1$ with $kn(P)=1, 2$

$\forall n>2$: positive number
$\exists P$: projection of $\mathbf{S}^1$ with $kn(P)=n$

ex.

P:



$kn(P)=n$

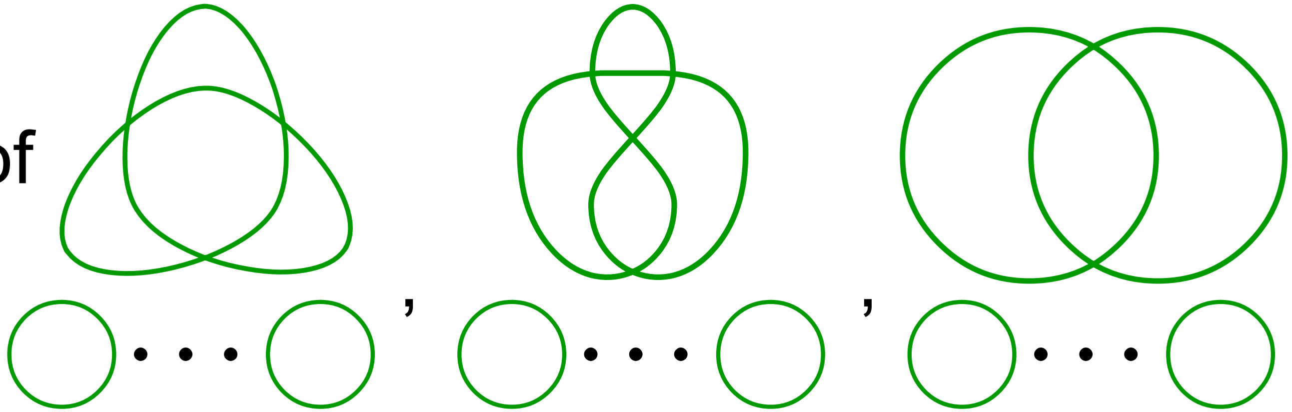
3.5 Knotting number

Theorem 4

P : projection of $\mathbf{S}^1 \amalg \mathbf{S}^1 \amalg \cdots \amalg \mathbf{S}^1$ with $kn(P) = p(P)$

where $p(P)$: the cardinality of the set of pre-crossings of P

$\Leftrightarrow$ P is one of

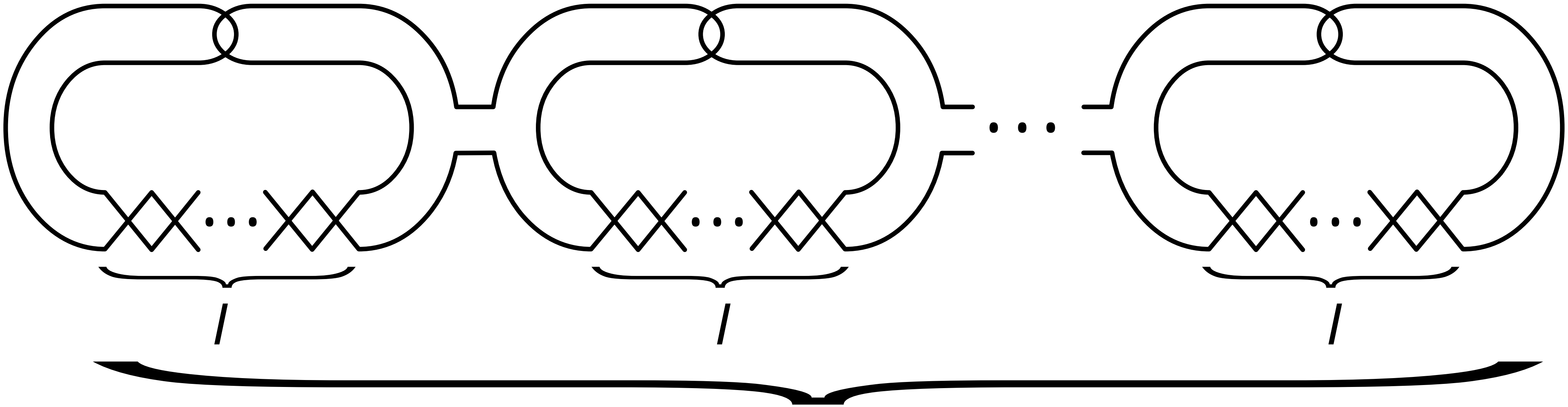


3.6 Trivializing number and Knotting number

Proposition 5

$\forall m$: even positive number, $\forall n$: positive number
$\exists P$: projection of $\mathbf{S}^1$ with $tr(P) = m$ and $kn(P) = n$

ex.



$\frac{m}{2}$

where $l = \left\lfloor \frac{n}{2} \right\rfloor + 3$

4.1 Application

P : projection of a graph

How many diagrams obtained from P represent trivial spatial graphs (resp. nontrivial spatial graphs)?

$n_{\text{tri}}(P) := \#\{D \mid D\text{: diagram obtained from } P \text{ s.t. } D \text{ represents a trivial spatial graph}\}$

$n_{\text{nontri}}(P) := \#\{D \mid D\text{: diagram obtained from } P \text{ s.t. } D \text{ represents a nontrivial spatial graph}\}$

Proposition 6

$$tr(P) \neq 0 \Rightarrow n_{\text{tri}}(P) \geqq 2^{p(P)-tr(P)+1}$$

$$kn(P) \neq 0 \Rightarrow n_{\text{nontri}}(P) \geqq 2^{p(P)-kn(P)+1}$$