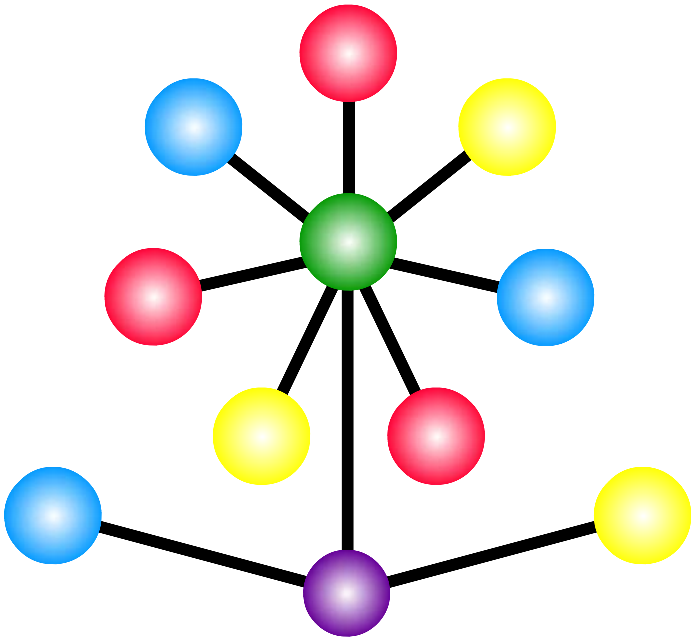
An elementary set for double-handcuff graph projections



早稲田大学大学院
教育学研究科

花木　良

1.1 Definitions

G : finite graph

f is a spatial embedding of G

$\overset{def.}{\Longleftrightarrow}$ $f: G \to R^3$: embedding

We call $f(G)$ a spatial graph

f, f' : spatial embeddings of G

f and f' are equivalent ($f \sim f'$)

$\overset{def.}{\Longleftrightarrow}$ $^\exists h: R^3 \to R^3$: (possibly orientation reversing) self-homeomorphism s.t. $h(f(G))=f'(G)$

1.1 Definitions

f is trivial

$\overset{def.}{\Leftrightarrow}$ ${}^{\exists}f' \sim f$ s.t. $f'(G) \subset R^2 \times \{0\} \subset R^3$

1.2 Definitions

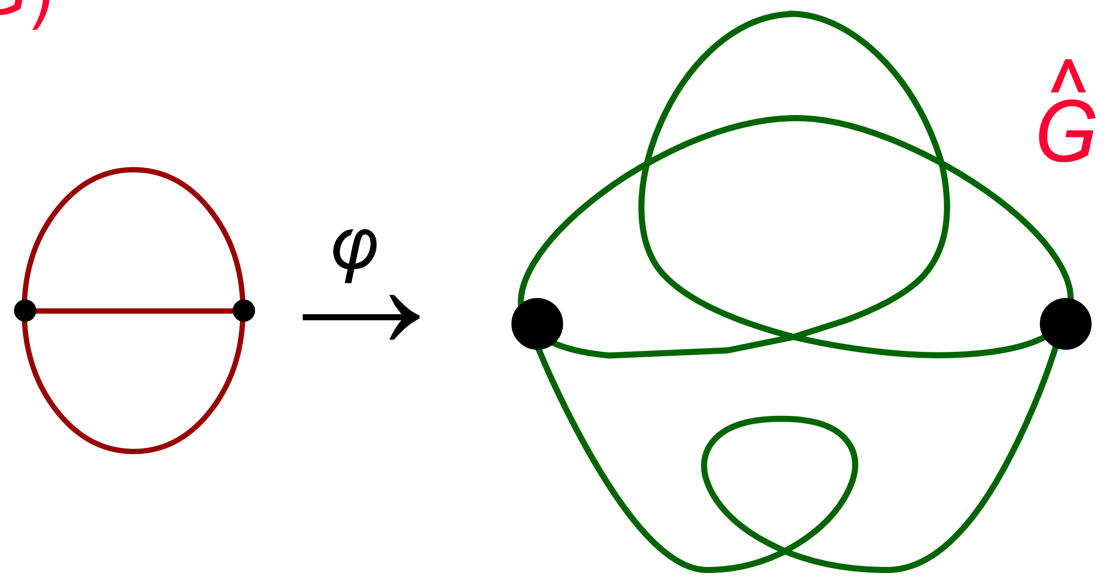
$\varphi : G \rightarrow R^2$: continuous map

φ is a projection of G

$\overset{def.}{\Leftrightarrow}$ multiple points of φ are only finitely many transversal double points of edges

We call the image of a projection a regular projection

and denote it by $\hat{G} = \varphi(G)$



1.2 Definitions

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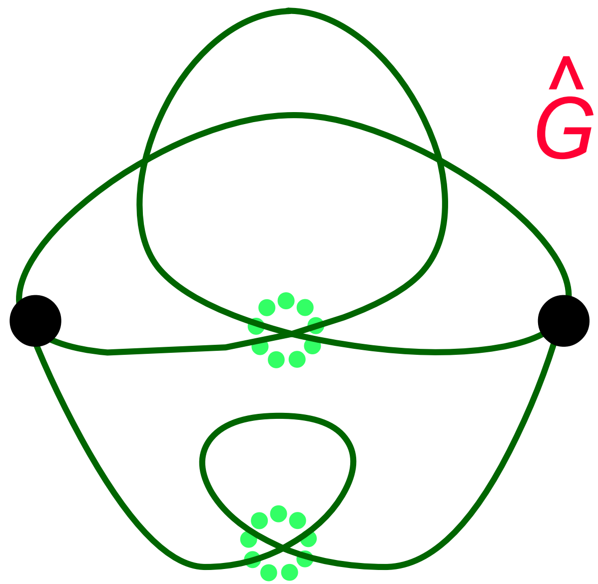


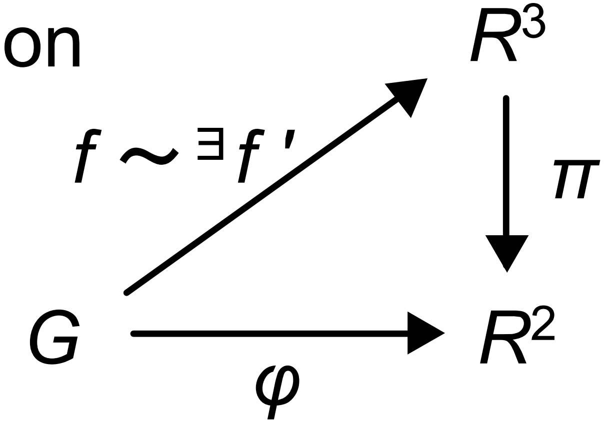
A double point of a regular projection is called a crossing

In particular,
a crossing point c is a self-crossing
$\overset{def.}{\Leftrightarrow}$ $\varphi^{-1}(c) \subset e$, where e is an edge of G

1.2 Definitions

φ is a projection of an embedding f

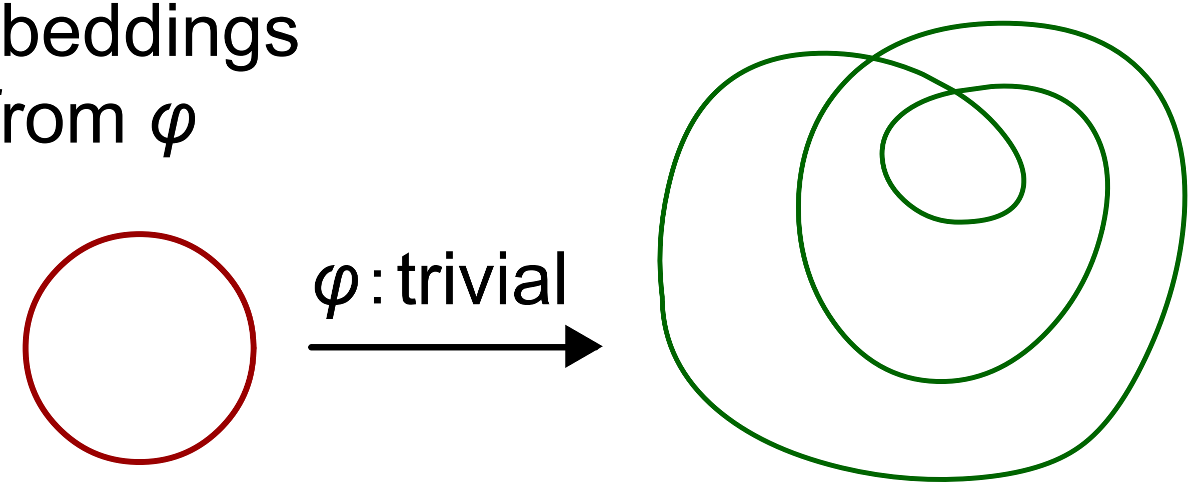
$\overset{def.}{\Leftrightarrow}$ $\exists f' \sim f$ s.t. $\varphi = \pi \circ f'$
where $\pi : R^3 \to R^2$ is a natural projction



We say f is obtained from φ

projection φ is trivial

$\overset{def.}{\Leftrightarrow}$ only trivial embeddings
are obtained from φ



1.3 Definitions

$\mathcal{E}$: a set of nontrivial embeddings of G

$\mathcal{E}$ is an elementary set of G

$\overset{def.}{\Longleftrightarrow}$ $\mathcal{E}$ satisfies the following ;

$\left.\begin{array}{l} {}^{\forall}\varphi \text{ : nontrivial projection of } G \\ \varphi \text{ is a projection of at least one element of } \mathcal{E} \end{array}\right\} (\ast)$

and ${}^{\forall}\mathcal{F} \subsetneq \mathcal{E}$ does not satisfy $(\ast)$

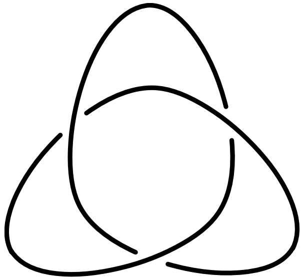
$elm(G) := \min\{\#\mathcal{E} \mid \mathcal{E}$ is an elementary set of $G\}$

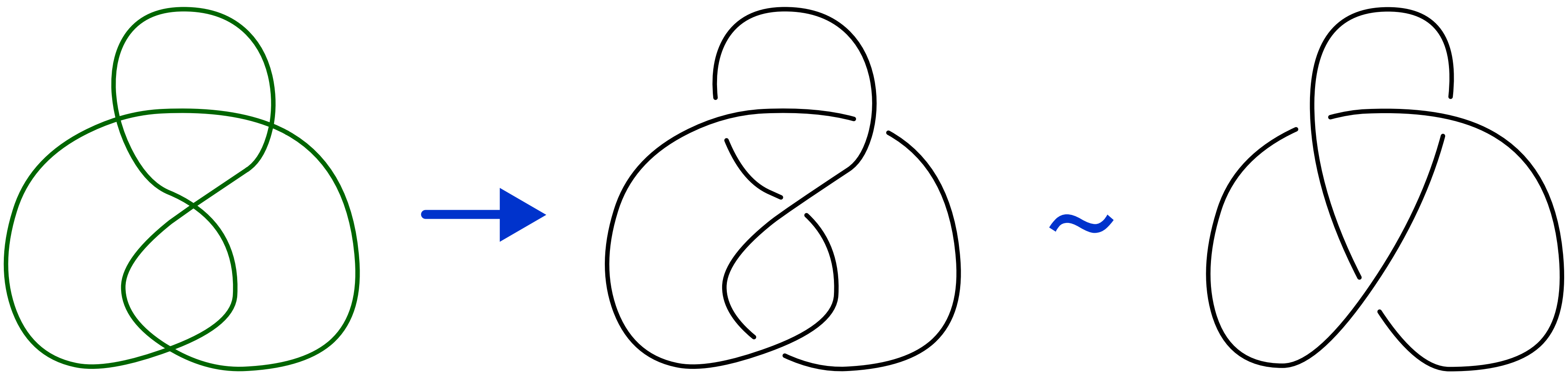
We call $elm(G)$ the elementary number of G.

2.1 Example 1 ($\mathbf{S}^1$)

[Taniyama 1989]

$G = \mathbf{S}^1$

$\mathcal{E} = \{$  $\}$ and $elm(G) = 1$



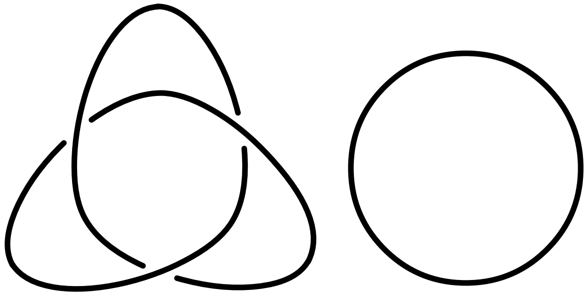
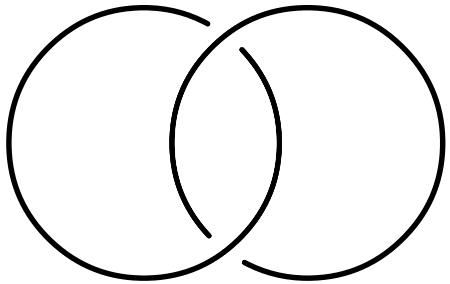
φ

f : spatial embedding obtained from φ

2.2 Example 2 ($\mathbf{S}^1 \amalg \mathbf{S}^1$)

[Taniyama 1989]

$G = \mathbf{S}^1 \amalg \mathbf{S}^1$

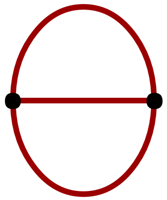
$\mathcal{E} = \{$  ,  $\}$

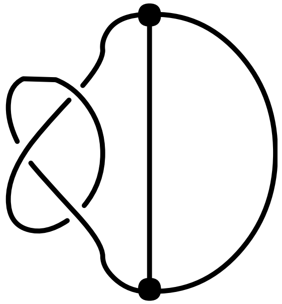
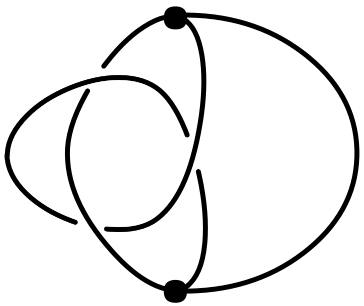
and $elm(G) = 2$

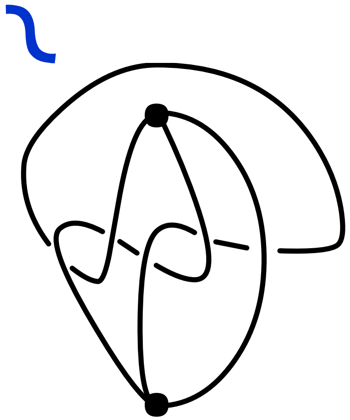
2.3 Example 3-1 (θ-curve)

[Kinoshita-Mikasa 1993], [Huh-Jin-Oh 2002]

G : θ-curve



$\mathcal{E} = \{$  ,  ,  $\}$

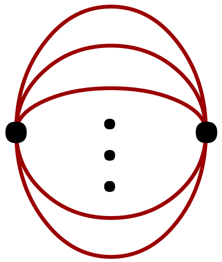


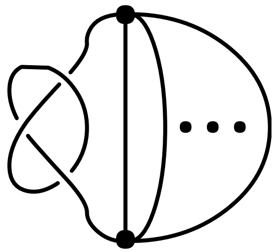
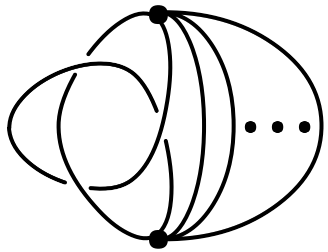
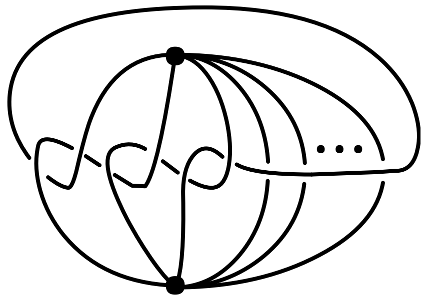
and $elm(G) = 3$

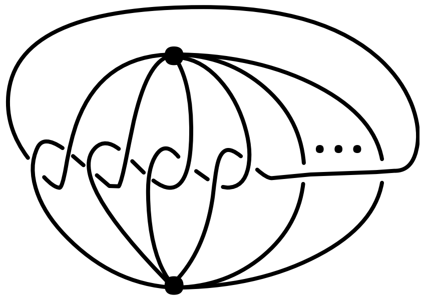
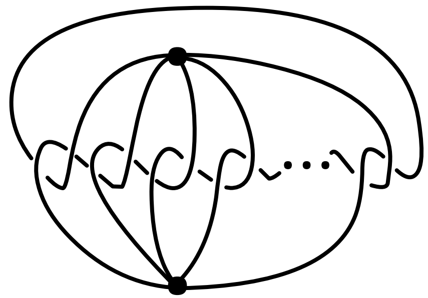
2.4 Example 3-2 (θ_n-curve)

[Huh-Jin-Oh 2002]

G : θ_n -curve



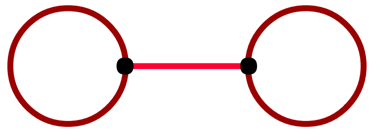
$\mathcal{E} = \{$  ,  , 

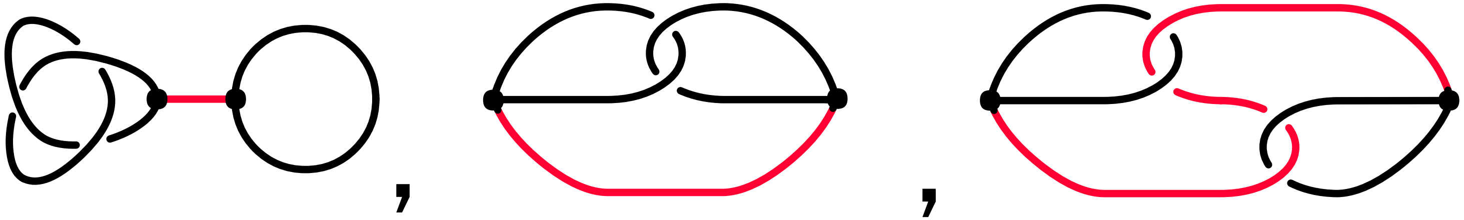
,  , $\cdots$,  $\}$

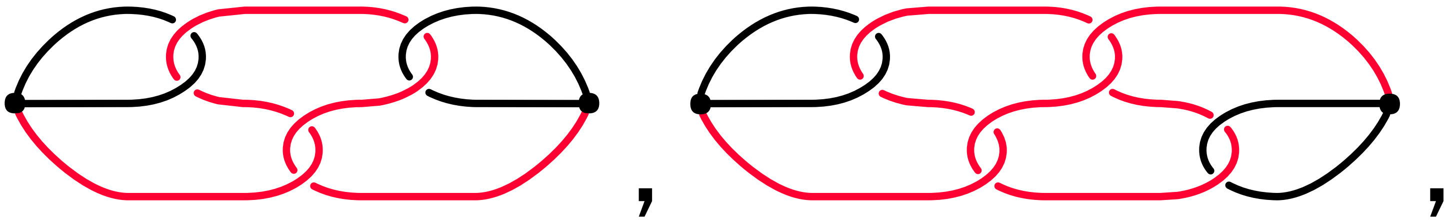
and $elm(G) = n$

2.5 Example 4-1 (handcuff graph)

[Taniyama-Yoshioka 1998]

G : handcuff graph 

$\mathcal{E} = \{$ 

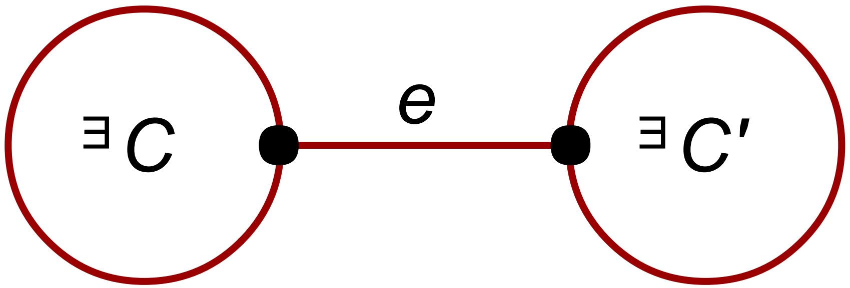
 $, \cdots \}$

and $elm(G) = \infty$

2.6 Example 4-2

[Taniyama-Yoshioka 1998]

G : connected planar graph with a cut edge e
s.t. each component of $G-\mathrm{int}\, e$ contains cycles



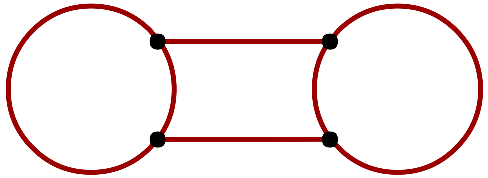
C, C': cycles

$elm(G)=\infty$

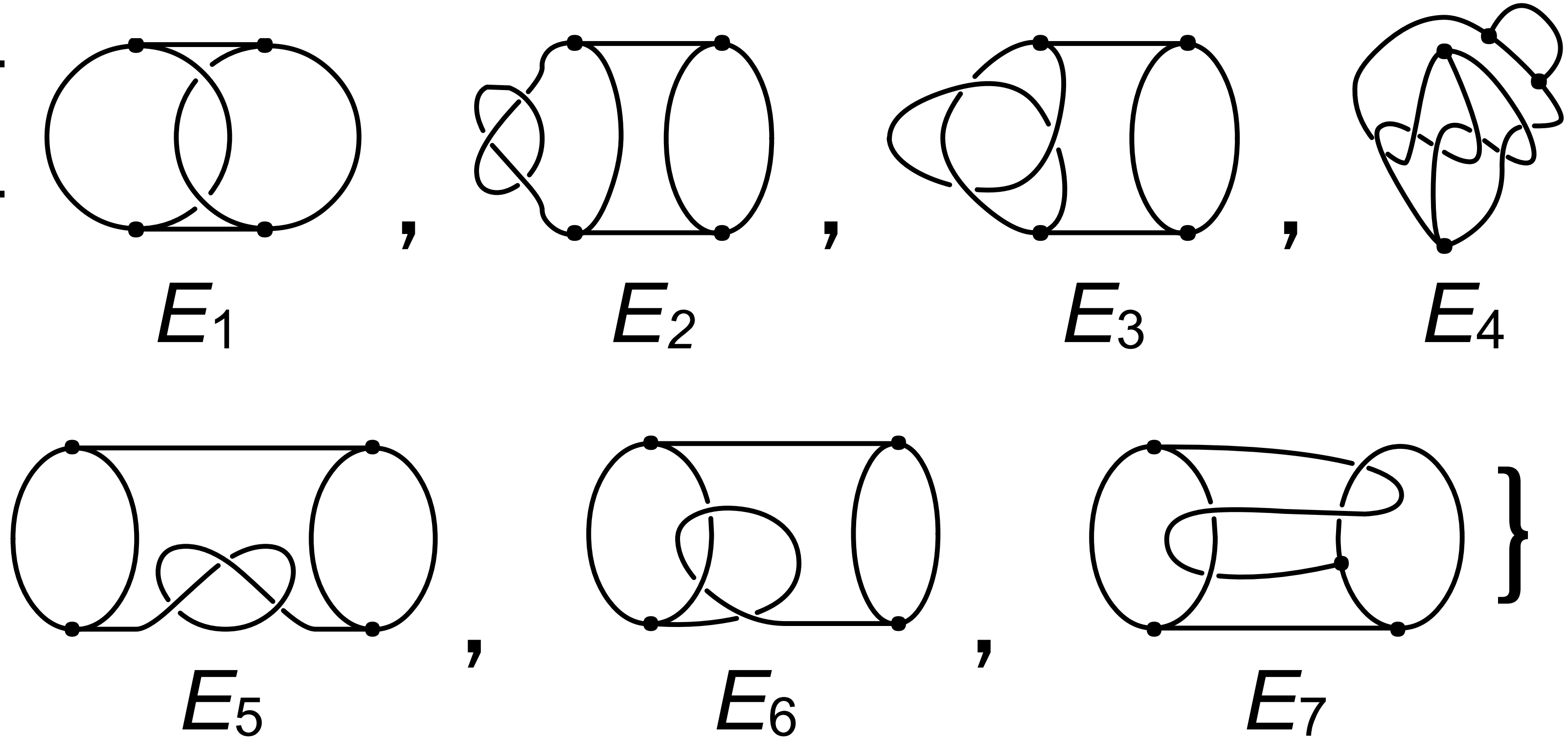
3.1 Main Theorem

Theorem 1 [Hanaki]

H : double-handcuff graph



$\mathcal{E} = \{$



$\}$

and $elm(H) = 7$

3.2 Preparations

c : crossing, P : regular projection

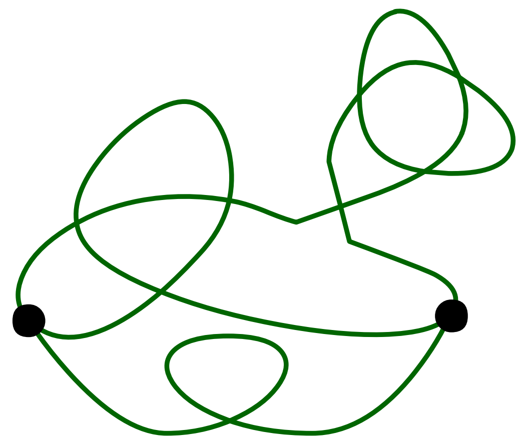
c of P is nugatory

$\overset{def.}{\Longleftrightarrow}$ $\omega(P-c)>\omega(P)$

where ω is the number of connected components

P is reduced

$\overset{def.}{\Longleftrightarrow}$ P has no nugatory crossing



3.2 Preparations

Proposition 1

$^\forall P$: regular projection with nugatory crossings

$^\exists P'$: reduced regular projection s.t. EMB(P')＝EMB(P)

where EMB(P) is the set of all embeddings obtained from P

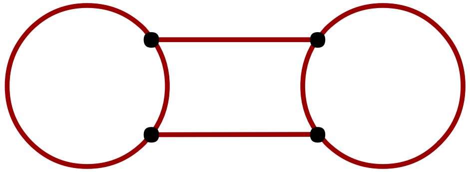
Proof

We can assume that

a regular projection P is a reduced regular projection.

3.3 Outline of Proof

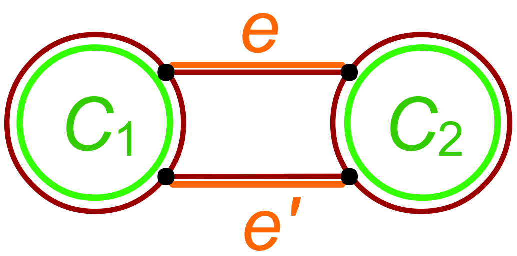
H : double-handcuff graph



By applying Lemmas 1, 2, 3, 4,
$^\forall \varphi$: nontrivial projection of H
φ is a projection of at least one element of $\mathcal{E}$,
that is, $elm(H) \leqq 7$.

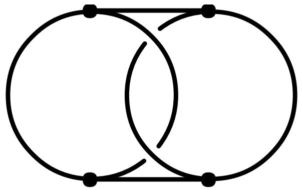
By applying Lemma 5, $elm(H) \geqq 7$.

3.4 Lemma 1

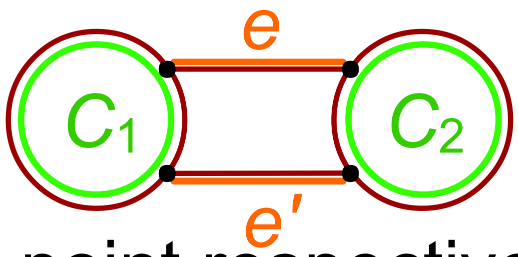


Lemma 1

$\varphi(C_1) \cap \varphi(C_2) \neq \emptyset$. ⇒ φ is a projection of E_1.



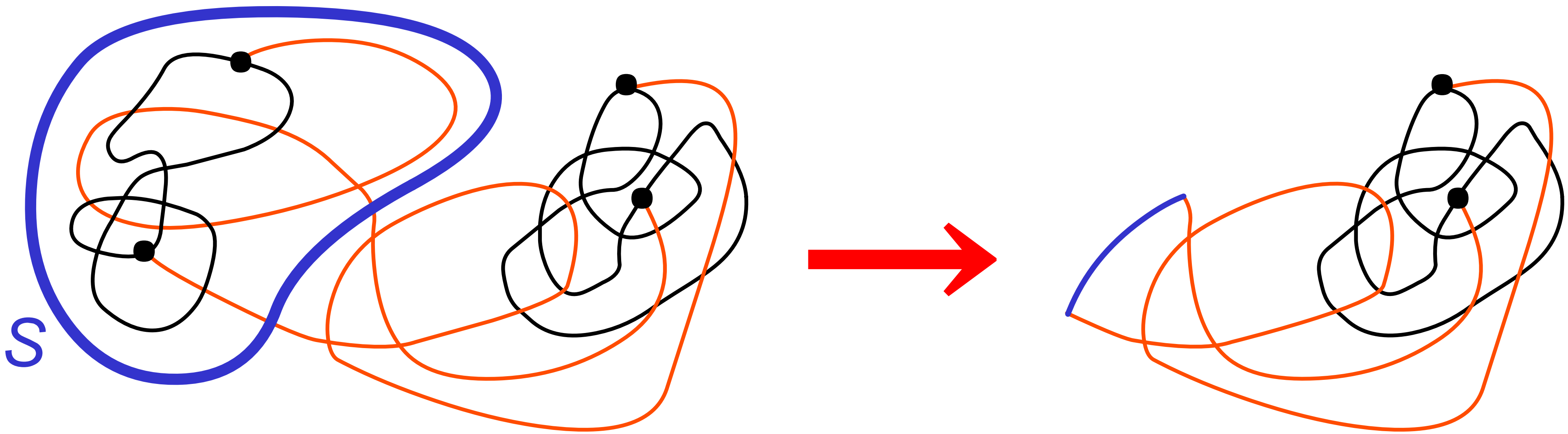
3.5 Lemma 2



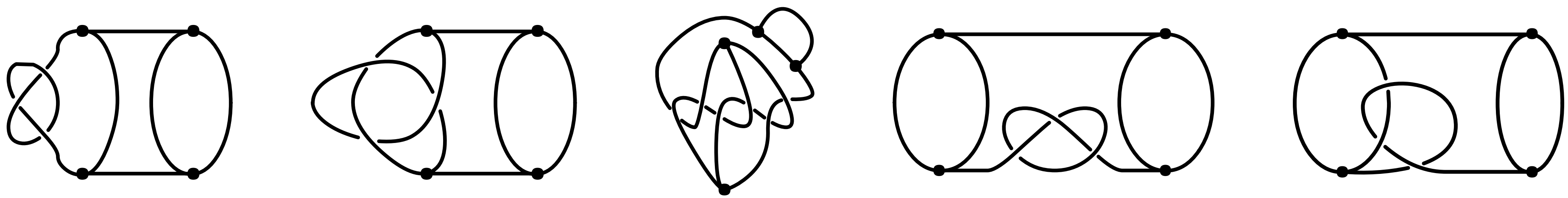
Lemma 2

$^{\exists} S$: circle on R^2 s.t.

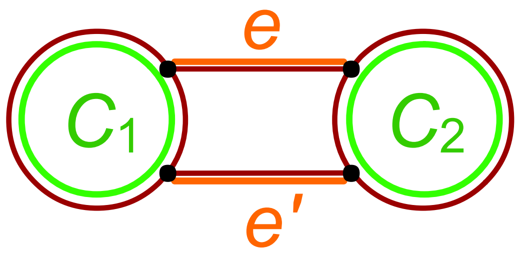
S meets $\varphi(e)$ and $\varphi(e')$ transversally at one point respectively



$\Rightarrow$ φ is a projection of E_2, E_3, E_4, E_5 or E_6.

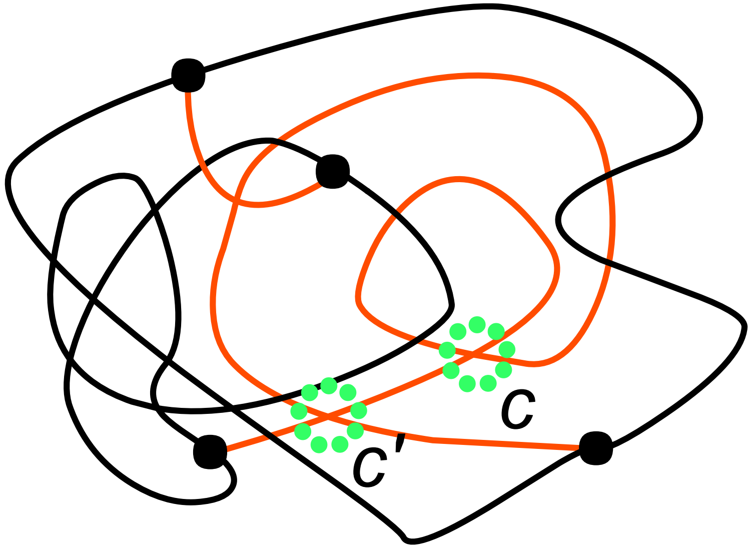


3.6 Lemma 3



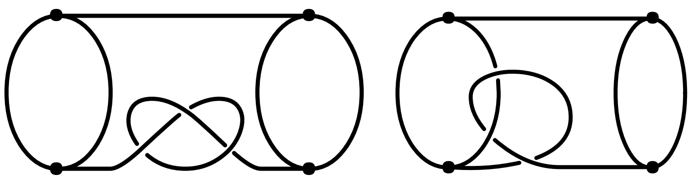
Lemma 3

$\varphi(e)$ or $\varphi(e')$ has self-crossings.

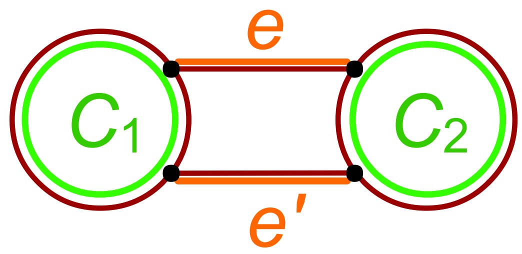


c, c': self-crossings

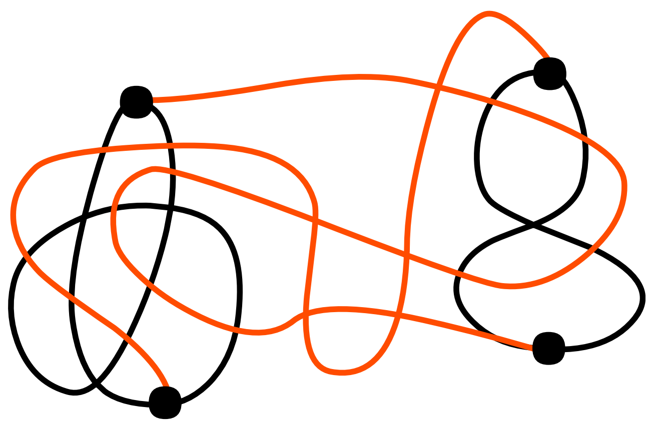
$\Rightarrow$ φ is a projection of E_5 or E_6.



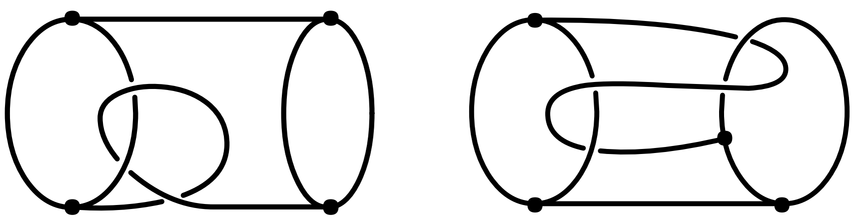
3.7 Lemma 4



Lemma 4
$\varphi(C_1) \cap \varphi(C_2) = \emptyset$,
$\nexists S$: circle on R^2 s.t.
S meets $\varphi(e)$ and $\varphi(e')$ at one point respetively
, and $\varphi(e)$ and $\varphi(e')$ has no self-crossing



$\Rightarrow$ φ is a projection of E_6 or E_7.



3.8 Lemma 5

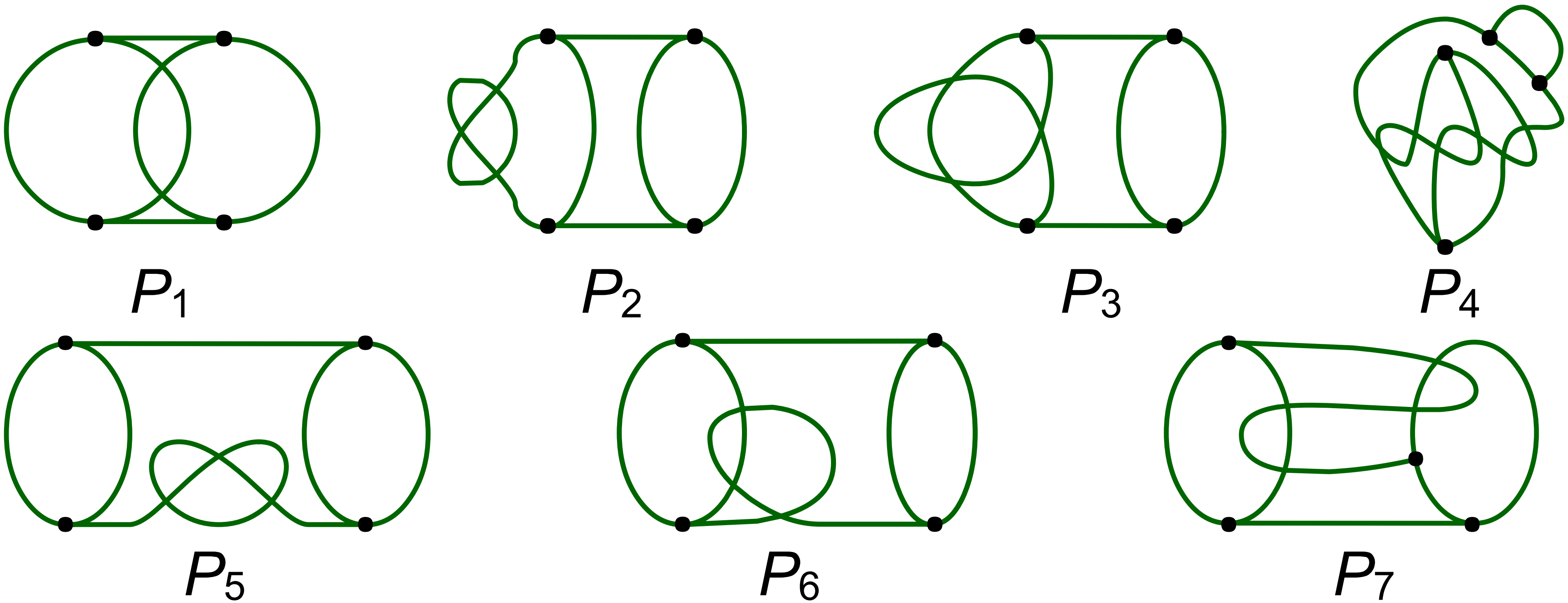
Lemma 5 $elm(H) \geqq 7$.

Proof (sketch)

Let P_i be a regular projection ($i = 1, 2, \ldots, 7$).

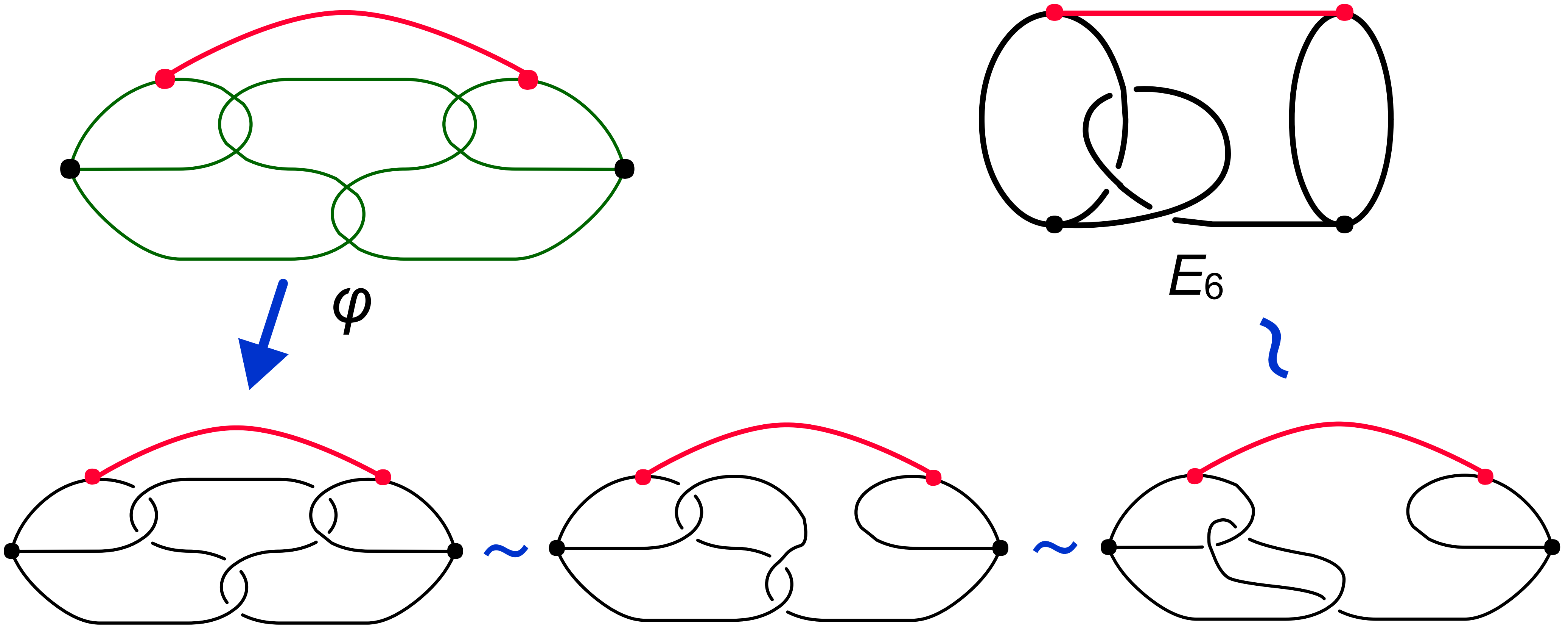
NE(P_i) : the set of all nontrivial embeddings obtained from P_i

Then we show that NE(P_i) ∩ NE(P_j) = ∅ ($i \neq j$).



3.9 Why is *elm*(*H*) finite?

Though the elementary number of handcuff graph is infinite,
the elementary number of double-handcuff graph is finite.



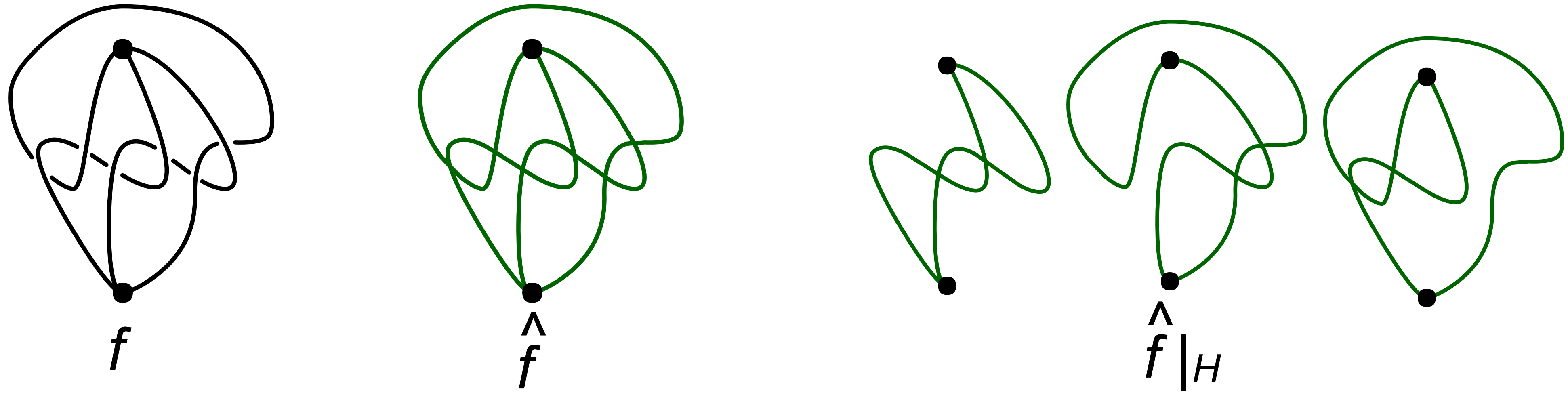
f : spatial embedding
obtained from φ

4.1 Remark (Preparations)

G : graph
f : nontrivial spatial embedding of G
f is strongly almost trivial.

$\overset{def.}{\Leftrightarrow}$ $^{\exists}\hat{f}$: projection of f
s.t. $^{\forall}H \subsetneq G$: proper subgraph, $\hat{f}|_H$ is trivial

Ex. θ-curve has strongly alomost trivial embeddings.



4.1 Remark (Preparations)

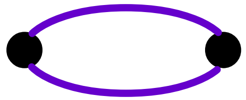
Theorem 2 [Huh-Oh, 2003]

G : planar graph s.t. $^{\forall} v \in V(G)$, v is not a cut vertex

$d(v) \geqq 3$ where $d(v)$ is the degree of v

G satisfies the followings

(1) G has no multiple edge.

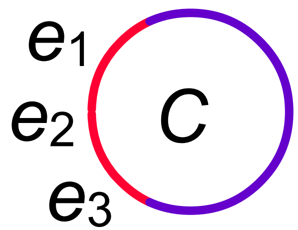


(2) $^{\forall} e_1, e_2 \in E(G)$ s.t. $e_1 \cap e_2 = \emptyset$,

$^{\exists} C_1, C_2$: disjoint cycles s.t. $e_1 \in E(C_1)$, $e_2 \in E(C_2)$

(3) $^{\forall} e_1, e_2, e_3 \in E(G)$ s.t. $e_1 \cup e_2 \cup e_3$ is homeo. to a path,

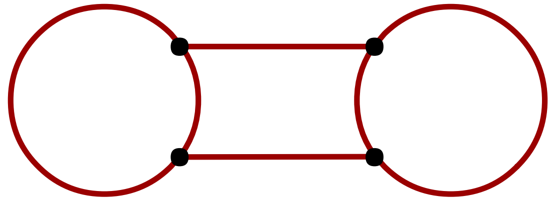
$^{\exists} C$: cycle s.t. $e_1, e_2, e_3 \in E(C)$



$\Rightarrow$ G has no strongly almost trivial embedding.

4.2 Remark

Double-handcuff graph H does not satisfy



(1) G has no multiple edge.

(2) $^\forall e_1, e_2 \in E(G)$ s.t. $e_1 \cap e_2 = \emptyset$,

$^\exists C_1, C_2$: disjoint cycles s.t. $e_1 \in E(C_1)$, $e_2 \in E(C_2)$

in Theorem 2.

However, the following Corollary 1 holds.

Corollary 1 [Hanaki]

H : double-handcuff graph

H has no strongly alomost trivial embedding.

4.2 Remark

Lemma 6

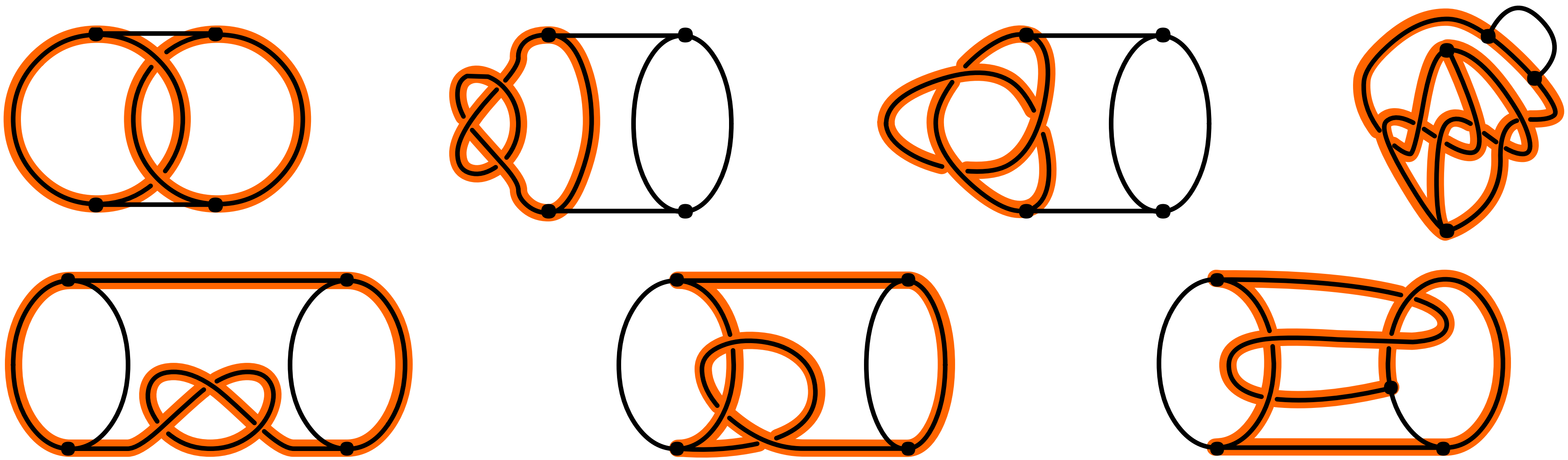
H : double-handcuff graph

$\mathcal{E}$: the elementary set in Theorem 1

For each $f \in \mathcal{E}$, $^{\exists}H' \subsetneq H$: proper subgraph of H

s.t. $f|H'$ is a nontrivial embedding

Proof



4.2 Remark

Corollary 1 [Hanaki]

H : double-handcuff graph

H has no strongly alomost trivial embedding.

Proof

f : nontrivial embedding of H

$^{\forall}\hat{f}$: projection of f Here $\hat{f}$ is nontrivial.

By Theorem 1, $^{\exists}g \in \mathcal{E}$ s.t. g is obtained from $\hat{f}$

By Lemma 6, $^{\exists}H' \subsetneq H$: proper subgraph of H

s.t. $g\,|H'$ is a nontrivial embedding

Hence $\hat{f}\,|H'$ is nontrivial.